Signature Optimization for CDMA With Limited Feedback

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***Abstract*—We study the performance of joint signature-receiver optimization for direct-sequence code-division multiple access (DS-CDMA) with limited feedback. The receiver for a particular user selects the signature from a signature codebook, and relays the corresponding B index bits to the transmitter over a noiseless channel. We study the performance of a random vector quantization (RVQ) scheme in which the codebook entries are independent and isotropically distributed. Assuming the interfering signatures are independent, and have independent and identically distributed (i.i.d.) elements, we evaluate the received signal-to-interference plus noise ratio (SINR) in the large system limit as the number of users, processing gain, and feedback bits B all tend to infinity with fixed ratios. This SINR is evaluated for both the matched filter and linear minimum mean-squared error (MMSE) receivers. Furthermore, we show that this large system SINR is the maximum that can be achieved over any sequence of codebooks. Numerical results show that with the MMSE receiver, one feedback bit per signature coefficient achieves close to single-user performance. We also consider a less complex and suboptimal reduced-rank signature optimization scheme in which the user's signature is constrained to lie in a lower dimensional subspace. The optimal subspace coefficients are scalar-quantized and relayed to the transmitter. The large system performance of the quantized reduced-rank scheme can be approximated, and numerical results show that it performs in the vicinity of the RVQ bound. Finally, we extend our analysis to the scenario in which a subset of users optimize their signatures in the presence of random interference.**

***Index Terms*—Asymptotics, code-division multiple access (CDMA), feedback, large system analysis, reduced-rank signature, signature optimization, vector quantization.**

I. Introduction

INTERFERENCE poses a major limitation on the performance of direct-sequence code-division multiple access (DS-CDMA). It has been recognized that in addition to interference suppression at the receiver, it is also possible to avoid interference by optimizing the signature sequences. Joint signature–receiver optimization with both ideal and frequency-selective channels has been discussed in [1]–[16]. For the most part, those studies assume that both the mobiles and base station have perfect knowledge of the signatures and the channel. Of course, the performance generally degrades when partial, or limited channel information is available. Here we are interested in the performance degradation due to *limited* feedback, i.e., the receiver can relay only a finite number of bits back to the transmitter. Our objective is to characterize the performance as a function of the amount of feedback allowed.

We analyze the performance of signature optimization with limited feedback for a CDMA model with additive white Gaussian noise and perfect power control (no fading). Both matched filter and linear minimum mean-squared error (MMSE) receivers are considered, and the receiver is assumed to have perfect knowledge of the interference-plus-noise covariance matrix. This implies that the receiver can compare the received signal-to-interference-plus noise ratio (SINR) associated with different signatures, and can compute the optimal signature for the corresponding user. (In practice, both the covariance matrix and optimal signature can be estimated with a training sequence.) The linear receivers considered have low complexity, and are optimal with unlimited feedback. We start by assuming that a single user optimizes his signature in the presence of fixed interfering signatures, which are independent, and contain independent and identically distributed (i.i.d.) elements. This scenario may correspond to a peer-to-peer network in which the transmitter adapts its signature to avoid interference from other stationary users.

Given B feedback bits, the receiver can select a signature from a signature *codebook* containing 2^B signatures. The selected signature maximizes the received SINR. The problem, then, is how to construct the codebook to maximize the received SINR averaged over the signatures. This appears to be difficult for a finite-size system, i.e., with finite number of users K, and processing gain N. We therefore instead consider a large system limit in which K, N, and B all tend to infinity with fixed ratios K/N, which is the system load, and B/N, which is the number of feedback bits per signature element.[1]

We evaluate the large system SINR of a *random vector quantization* (RVQ) scheme in which the signatures in the codebook are independent and isotropically distributed. Furthermore, for both the matched filter and MMSE receivers we show that this limiting SINR is the maximum that can be achieved over any sequence of signature codebooks. This result relies on the fact that in the presence of fixed, i.i.d. interfering signatures the optimal signature, namely, the eigenvector of the interference-plus-noise covariance matrix corresponding to the smallest eigenvalue, is isotropically distributed in the large system limit (e.g., see [17]). Numerical results show that the large system RVQ results accurately predict the performance of finite-size systems. The numerical results also show that with the matched-filter receiver,

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[1]Additional feedback is needed to specify the channel gain for power control and adaptive modulation. However, since that feedback does not scale with the system size, it becomes negligible in the large system limit.

one feedback bit per signature coefficient can typically achieve an SINR within 1 dB of the single-user bound. The MMSE receiver requires significantly less feedback to achieve the same performance.

A major drawback of RVQ is computational complexity. Namely, in general, the optimal quantized signature must be found by exhaustive search, and the number of entries in the codebook increases exponentially with the number of feedback bits. We therefore consider a scalar-quantized version of the *reduced-rank* signature scheme introduced in [15]. A reduced-rank signature is constrained to lie in a D-dimensional random subspace $(D \leq N)$ known to both the receiver and transmitter. The D subspace coefficients are optimized, and each is quantized with the same scalar quantizer. For a fixed number of feedback bits the dimension, or rank D can be optimized. Namely, as D increases, there are more degrees of freedom with which to avoid interference, but also more coefficients to quantize, and hence more performance degradation due to quantization error. The large system performance of reduced-rank signature optimization is evaluated both with and without scalar quantization. Numerical results show that this scheme (with optimized rank) typically performs within 1 dB of the RVQ bound.

We also extend our results to the scenario in which a group of users optimizes their signatures in the presence of fixed, random interference. Namely, we evaluate the large system performance of RVQ, and again compare with the performance of reduced-rank signatures.

Related work on joint transmitter–receiver optimization with limited feedback has been presented in [16], [18]–[22]. (See also [23]–[25], which consider feedback of time-varying channel statistics without an explicit constraint on the feedback rate.) With the exception of [16, Ch. 7], which evaluates the performance of scalar quantization schemes for CDMA via simulation, that work is motivated by multiantenna systems. The feedback is therefore used to specify spatial signatures, instead of temporal signatures. Our large system approach differs from the approach taken in those references, and allows us to compute an explicit upper bound on the performance of any signature quantization scheme. Application of this approach to multiple-input/multiple-output channels has been recently presented in [26], [27].

The rest of the paper is organized as follows. Section II introduces the system model, and Section III describes RVQ and characterizes its performance in the large system limit. Section IV discusses reduced-rank signature optimization with scalar quantization. The performance of group signature optimization with limited feedback is characterized in Sections V, and Section VI concludes the paper.

II. System Model

We consider a synchronous discrete baseband CDMA model with K users and processing gain N. The $N \times 1$ received vector for a particular symbol is given by

$$\boldsymbol{r} = \sum_{k=1}^{K} b_k \boldsymbol{s}_k + \boldsymbol{n} \tag{1}$$

where $\boldsymbol{s}_k$ is the $N \times 1$ signature for user k, b_k is the transmitted symbol for user k, and $\boldsymbol{n}$ is the additive white Gaussian noise vector with covariance matrix $\sigma_n^2 \boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix. We consider an ideal model to simplify the performance evaluation. The signature optimization methods, to be described, apply directly to an asynchronous model, and are easily extended to account for multipath, assuming the receiver knows (or estimates) the channel.

The performance of each user depends on the assigned signature set $\{\boldsymbol{s}_k\}$. The users and receivers can jointly adapt and update the signatures via a feedback channel. We start by assuming that a *single* user, corresponding to $k = 1$, optimizes his signature in the presence of fixed interfering signatures, corresponding to $\boldsymbol{s}_2, \ldots, \boldsymbol{s}_K$. This is motivated by a peer-to-peer scenario in which a user attempts to avoid interference from other stationary random users.[2] The interfering signatures are assumed to be independent, and to contain i.i.d. elements with zero mean and variance $1/N$, so that $E[\|\boldsymbol{s}_k\|^2] = 1$ and $\|\boldsymbol{s}_k\|^2 \to 1$ almost surely as $N \to \infty$ for each $k = 1, \ldots, K$. Power variations across users can also be incorporated in the following analysis; however, because of the additional complications that arise, here we focus on the equal power scenario. The performance of group signature optimization with limited feedback, motivated by the multiple-access channel, is addressed in Section V.

We assume that user 1 has a linear receiver, consisting of the $N \times 1$ vector $\boldsymbol{c}_1$. The performance measure is the SINR at the output of $\boldsymbol{c}_1$, given by

$$\beta_{\boldsymbol{c}_1} = \frac{|\boldsymbol{c}_1^\dagger \boldsymbol{s}_1|^2}{\boldsymbol{c}_1^\dagger \boldsymbol{R}_1 \boldsymbol{c}_1} \tag{2}$$

where $\dagger$ denotes Hermitian transpose and $\boldsymbol{R}_1$ is the interference-plus-noise covariance matrix given by

$$\boldsymbol{R}_1 = \boldsymbol{S}_1 \boldsymbol{S}_1^\dagger + \sigma_n^2 \boldsymbol{I} \tag{3}$$

where $\boldsymbol{S}_1$ is the $N \times (K-1)$ signature matrix whose columns are the interfering signatures $\{\boldsymbol{s}_2, \ldots, \boldsymbol{s}_K\}$. The matrix $\boldsymbol{R}_1$ is known at the receiver, but not the transmitter.

We will consider two linear receivers for user 1, the matched filter, and the linear MMSE receiver. The matched filter is given by

$$\boldsymbol{c}_1 = \boldsymbol{s}_1 \tag{4}$$

with corresponding output SINR

$$\beta_{\mathsf{mf}} = \frac{1}{\sigma_n^2 + \boldsymbol{s}_1^\dagger \boldsymbol{S}_1 \boldsymbol{S}_1^\dagger \boldsymbol{s}_1}. \tag{5}$$

The matched filter is relatively simple to analyze, and is robust in an adaptive mode with limited training for signature estimation [15]. The MMSE receiver maximizes the output SINR, and is given by

$$\boldsymbol{c}_1 = \boldsymbol{R}^{-1} \boldsymbol{s}_1 \tag{6}$$

[2]In the peer-to-peer scenario, the receivers are not colocated, so that $\boldsymbol{r}$ depends on the user index k.

where the received covariance matrix $\boldsymbol{R} = \boldsymbol{S}\boldsymbol{S}^\dagger + \sigma_n^2\boldsymbol{I}$, and $\boldsymbol{S}$ is the $N \times K$ signature matrix containing all signatures. Substituting (6) and (3) into (2) gives the output SINR

$$\beta_{\mathsf{mmse}} = \boldsymbol{s}_1^\dagger \boldsymbol{R}_1^{-1}\boldsymbol{s}_1. \tag{7}$$

III. Random Vector Quantization

Ideally, the signature for user 1, $\boldsymbol{s}_1$, should be chosen to maximize the received SINR. It is easily shown that for a fixed $\boldsymbol{c}_1$, the optimal signature $\boldsymbol{s}_1 = \boldsymbol{c}_1/\|\boldsymbol{c}_1\|$. Rewriting the MMSE receiver in (6) as $\boldsymbol{c}_1 = \kappa \boldsymbol{R}_1^{-1}\boldsymbol{s}_1$, where κ is a constant,[3] and substituting in the preceding expression for $\boldsymbol{s}_1$ shows that the optimal signature $\boldsymbol{s}_1$ is an eigenvector of $\boldsymbol{R}_1$. Furthermore, (7) implies that this eigenvector corresponds to the minimum eigenvalue. That is, the optimal signature lies in the direction of least interference plus noise [11], [16]. Furthermore, with the optimized signature, the MMSE and matched-filter receivers are the same.

In practice, the receiver must relay a quantized version of the optimal signature back to the transmitter. Since the receiver knows $\boldsymbol{R}_1$, it can compute the optimal signature; however, it is allowed to relay only a finite number of feedback bits B to the transmitter. We assume that the feedback does not incur any errors. In general, the receiver can choose the signature from a signature codebook containing 2^B signatures. This codebook is designed *a priori*, and is known at both the transmitter and receiver. The problem, then, is how to design the codebook to maximize the SINR, averaged over the random signatures, symbols, and noise. This can be interpreted as a vector quantization problem, except that the performance objective is received SINR, rather than the fidelity of the quantized signature.

We denote the signature codebook for user 1 as

$$\mathcal{V} = \{\boldsymbol{v}_j; 1 \le j \le 2^B\}. \tag{8}$$

The receiver selects the signature, which maximizes the SINR

$$\boldsymbol{s}_1 = \arg\max_{1\le j\le 2^B} \beta_{\boldsymbol{v}_j} \tag{9}$$

where $\beta_{\boldsymbol{v}_j}$ is the SINR resulting from signature $\boldsymbol{v}_j$. Given statistical properties of the interfering signatures, it is generally difficult to construct the optimal codebook and evaluate its performance for any particular set of processing gain, number of users, and feedback bits.[4] However, we will see that for the matched-filter receiver, a tight upper bound on the optimal performance can be derived in the large system limit in which $(N, K, B) \to \infty$ with fixed normalized load $\bar{K} = K/N$, and normalized feedback bits per dimension $\bar{B} = B/N$.

We illustrate the large system limit by first considering two extreme cases, no feedback ($\bar{B} = 0$) and unlimited feedback ($\bar{B} = \infty$). When $\bar{B} = 0$, the transmitter has no knowledge of the interference covariance matrix. Since for large N the interfering signatures are isotropically distributed, the transmitter also chooses $\boldsymbol{s}_1$ from an isotropic distribution. The SINR is then a random variable, which converges almost surely to a deterministic value in the large system limit. The asymptotic SINR at the output of the matched filter is given by [28]

$$\lim_{(N,K,B)\to\infty} \beta_{\mathsf{mf}} = \beta_{\mathsf{mf}}^\infty(\bar{B} = 0) = \frac{1}{\sigma_n^2 + \bar{K}}. \tag{10}$$

For the MMSE receiver, the asymptotic SINR with random signatures satisfies the fixed-point equation [29]

$$\begin{aligned}\lim_{(N,K,B)\to\infty} \beta_{\mathsf{mmse}} &= \beta_{\mathsf{mmse}}^\infty(\bar{B} = 0)\\ &= \frac{1}{\sigma_n^2 + \frac{\bar{K}}{1+\beta_{\mathsf{mmse}}^\infty(\bar{B}=0)}}.\end{aligned} \tag{11}$$

With unlimited feedback, the signature is the eigenvector of $\boldsymbol{R}_1$ corresponding to the minimum eigenvalue, and the SINRs for the matched filter and MMSE receivers are given by

$$\begin{aligned}\beta_{\mathsf{mf}}^\infty(\bar{B} = \infty) &= \beta_{\mathsf{mmse}}^\infty(\bar{B} = \infty)\\ &= \begin{cases} \frac{1}{\sigma_n^2}, & \bar{K} \le 1\\ \frac{1}{\sigma_n^2 + (\sqrt{\bar{K}}-1)^2}, & \bar{K} > 1.\end{cases}\end{aligned} \tag{12}$$

That is, when $\bar{K} \le 1$ the signature is orthogonal to the interfering signatures, giving single-user performance. When $\bar{K} > 1$, the SINR is the minimum eigenvalue of $\boldsymbol{R}_{-1}^{-1}$, which converges in the large system limit to the value shown.

As the size of the interference-plus-noise covariance matrix grows according to the large system limit, the eigenvectors become isotropically distributed [17]. This motivates an RVQ scheme in which the (random) codebook entries are independent and isotropically distributed. We will show that RVQ is optimal in the large system limit for both linear receivers, i.e., it achieves the maximum large system SINR.

A. Matched Filter

Maximizing the SINR with a matched filter, given by (5), is equivalent to minimizing the interference power, i.e.,

$$\boldsymbol{s}_1 = \arg\min_{\boldsymbol{v}_j\in\mathcal{V}} \left\{ X_j = \sum_{i=2}^{K} (\boldsymbol{v}_j^\dagger \boldsymbol{s}_i)^2 \right\} \tag{13}$$

where X_j is the interference power corresponding to signature $\boldsymbol{v}_j$. The minimum interference is therefore given by

$$W = \min\{X_1, X_2, \ldots, X_{2^B}\}. \tag{14}$$

We are interested in computing the average

$$E[W] = E_{\mathcal{V},\boldsymbol{S}_1}\left[\min_{1\le j\le 2^B}\{X_j\}\right] \tag{15}$$

i.e., the expectation is with respect to both the codebook and the set of interfering signatures.

For RVQ, the vectors $\{\boldsymbol{v}_j\}$ are i.i.d., hence the corresponding interference terms $\{X_j\}$ are also i.i.d., conditioned on the interfering signature matrix $\boldsymbol{S}_1$. Given the probability density func-

[3]This follows from (6) and the matrix inversion lemma.

[4]A Lloyd–Max algorithm is used to construct an optimal quantizer for a beamformer in [18], and a precoding matrix for a multiple-input/multiple-output channel in [21]. This scheme is quite complicated and the associated performance is difficult to evaluate.

tion (pdf) for X_j given $\boldsymbol{S}_1$, $f_{X|\boldsymbol{S}_1}(\cdot)$, and the cumulative distribution function (cdf) $F_{X|\boldsymbol{S}_1}(\cdot)$, the cdf for W given $\boldsymbol{S}_1$ can be expressed as

$$F_{W\,|\,\boldsymbol{S}_1}(x) = 1 - \left(1 - F_{X\,|\,\boldsymbol{S}_1}(x)\right)^{2^B} \tag{16}$$

and the expected interference is given by

$$E_{\mathcal{V}}[W\,|\,\boldsymbol{S}_1] = \int_0^\infty x 2^B \left(1 - F_{X|\boldsymbol{S}_1}(x)\right)^{2^B-1} f_{X\,|\,\boldsymbol{S}_1}(x)\,\mathrm{d}x. \tag{17}$$

Closed-form expressions for $f_{X|\boldsymbol{S}_1}(\cdot)$ and $F_{X|\boldsymbol{S}_1}(\cdot)$ are given in [30] (see (56) in Appendix A), and depend only on the eigenvalues of the interference covariance matrix $\boldsymbol{S}_1\boldsymbol{S}_1^\dagger$. To emphasize this property, in what follows we will write the conditional cdf $F_{X\,|\,\boldsymbol{S}_1}$ as $F_{X\,|\,\Lambda}$ where Λ denotes the set of eigenvalues of $\boldsymbol{S}_1\boldsymbol{S}_1^\dagger$. Evaluating (17) for finite (N, K, B) appears to be difficult, and must be done numerically. We therefore consider the large system limit $(N, K, B) \to \infty$. In this limit, the empirical distribution of eigenvalues in the set Λ converges to a deterministic distribution, independent of the particular realization of $\boldsymbol{S}_1$ [31].

As $(K, N) \to \infty$, it can be shown that each X_j converges to K/N in the mean-square sense. That is, $E_{\mathcal{V},\boldsymbol{S}_1}[X_j] \to \bar{K}$, and the variance goes to zero as $1/N$. Hence, for any fixed B, $W \to \bar{K}$. However, if we also let $B \to \infty$ with fixed B/N, then as N increases, W is the minimum of an exponentially increasing number of samples from the cdf $F_{X|\Lambda}(\cdot)$. In that case, W converges to a deterministic limit between zero, corresponding to $\bar{B} = \infty$, and $\bar{K}$, corresponding to $\bar{B} = 0$. This limit is given by the following theorem.

Theorem 1: As $(N, K, B) \to \infty$ with fixed ratios $\bar{K} = K/N \geq 0$ and $\bar{B} = B/N \geq 0$, the interference power W with RVQ converges in the mean-square sense to

$$W^\infty = \lim_{(N,K,B)\to\infty} F_{X\,|\,\Lambda}^{-1}\left(\frac{1}{2^B}\right). \tag{18}$$

Furthermore, $W^\infty \geq \bar{W}^\infty$, where $\bar{W}^\infty$ satisfies

$$(\bar{W}^\infty)^{\bar{K}} \mathrm{e}^{-\bar{W}^\infty} = \left(\frac{\bar{K}}{\mathrm{e}}\right)^{\bar{K}} 2^{-2\bar{B}} \tag{19}$$

The proof relies on the asymptotic theory of extreme order statistics [32] and is given in Appendix A. Note that for $\bar{K} \leq 1$, the bound is tight when $\bar{B} = 0$, in which case $\bar{W}^\infty = \bar{K}$, and when $\bar{B} \to \infty$, in which case $\bar{W}^\infty = 0$. The large system SINR for RVQ is therefore given by

$$\beta_{\mathsf{mf;rvq}}^\infty = \frac{1}{\sigma_n^2 + W^\infty} \leq \frac{1}{\sigma_n^2 + \bar{W}^\infty}. \tag{20}$$

The lower bound on the minimum interference power W^∞ in Theorem 1 is obtained by conditioning on the *codebook* $\mathcal{V}$ in (15), instead of conditioning on $\boldsymbol{S}_1$. In that case, $\boldsymbol{S}_1$ is random, and given $\boldsymbol{v}_j, \boldsymbol{v}_j^\dagger\boldsymbol{s}_i$ has a Gaussian distribution with zero mean and variance $1/N$. Therefore, the X_j's, conditioned on $\mathcal{V}$, are identically distributed with a Gamma pdf, which has mean $\bar{K}$ and variance $2\bar{K}/N$, and is given by

$$f_{X\,|\,\mathcal{V}}(x) = \frac{N}{2\Gamma\left(\frac{K}{2}\right)}\left(\frac{Nx}{2}\right)^{\frac{K-2}{2}} \mathrm{e}^{-\frac{Nx}{2}}, \qquad x \geq 0 \tag{21}$$

where $\Gamma(\cdot)$ is the complete Gamma function.

Evaluation of W^∞ is complicated by the fact that the X_j's, conditioned on $\mathcal{V}$, are *not* independent. (That is, they all depend on the same random matrix $\boldsymbol{S}_1$.) It is shown in the proof of Theorem 2 that replacing the X_j's with i.i.d. random variables gives a lower bound on the minimum interference power. Applying the theory of extreme order statistics in that case gives

$$\bar{W}^\infty = \lim_{(N,K,B)\to\infty} F_{X|\mathcal{V}}^{-1}\left(\frac{1}{2^B}\right)$$

which is shown to satisfy (19).

In general, we can consider an arbitrary sequence of signature codebooks $\{\mathcal{V}_N\}$ where $\mathcal{V}_N = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{2^B}\}$, and $\|\boldsymbol{v}_j\|^2 = 1$ for each j. Let $\beta_{\mathcal{V}_N}$ denote the corresponding SINR averaged over random interfering signatures.

Theorem 2: For any sequence of signature codebooks $\{\mathcal{V}_N\}$, $\limsup \beta_{\mathcal{V}_N} \leq \beta_{\mathsf{mf;rvq}}^\infty$.

The proof is given in Appendix B.

The theorem states that with a matched-filter receiver, RVQ is asymptotically optimal in the sense that it maximizes the large system SINR. Hence, the performance of RVQ serves as an upper bound on the large system performance of any quantization scheme. Note that RVQ is suboptimal for a finite-size system, since signatures in an optimal codebook should be orthogonal. (The RVQ signatures become orthogonal as $N \to \infty$.) Still, numerical comparisons with other optimized schemes [19] in the context of beamforming with limited feedback show that RVQ performance is indistinguishable from the optimized performance even for small systems [33].

B. MMSE Receiver

In this case, the receiver selects the signature from the codebook $\mathcal{V}$, which maximizes the corresponding SINR, i.e.,

$$\boldsymbol{s}_1 = \arg\max_{\boldsymbol{v}_j \in \mathcal{V}} \left(\beta_{\mathsf{mmse,j}} = \boldsymbol{v}_j^\dagger \boldsymbol{R}_1^{-1} \boldsymbol{v}_j\right) \tag{22}$$

where $\beta_{\mathsf{mmse,j}}$ denotes the SINR corresponding to $\boldsymbol{v}_j$. The SINR with RVQ is therefore,

$$\beta_{\mathsf{mmse;rvq}} = \max_{1\leq j\leq 2^B} \{\beta_{\mathsf{mmse,j}}\} \tag{23}$$

where the $\beta_{\mathsf{mmse;j}}$'s are independent, conditioned on $\boldsymbol{S}_1$, because the quantized signatures are independent. The SINR averaged over the RVQ codebook $\mathcal{V}$ is therefore,

$$E_{\mathcal{V}}[\beta_{\mathsf{mmse;rvq}}\,|\,\boldsymbol{S}_1] = \int_0^{\frac{1}{\sigma_n^2}} x 2^B [G_{\beta\,|\,\boldsymbol{S}_1}(x)]^{2^B-1} g_{\beta\,|\,\boldsymbol{S}_1}(x)\,\mathrm{d}x \tag{24}$$

where $G_{\beta\,|\,\boldsymbol{S}_1}$ and $g_{\beta\,|\,\boldsymbol{S}_1}$ are, respectively, the cdf and pdf for $\beta_{\mathsf{mmse,j}}$, given $\boldsymbol{S}_1$, and depend only on the eigenvalues of $\boldsymbol{R}_1^{-1}$ or equivalently, the eigenvalues of $\boldsymbol{S}_1\boldsymbol{S}_1^\dagger$. As for the matched-filter receiver, we again write $G_{\beta\,|\,\Lambda}$ instead of $G_{\beta\,|\,\boldsymbol{S}_1}$ to emphasize this fact.

In the large system limit, the SINR $\beta_{\mathsf{mmse;rvq}}$ converges to a deterministic limit $\beta^{\infty}_{\mathsf{mmse;rvq}}$, which is given in the following theorem.

Theorem 3: As $(N,K,B) \to \infty$ with fixed $\bar{K} \geq 0$ and $\bar{B} \geq 0$, the SINR $\beta_{\mathsf{mmse;rvq}}$ converges in the mean-square sense to

$$\beta^{\infty}_{\mathsf{mmse;rvq}} = \lim_{(K,N,B)\to\infty} G^{-1}_{\beta\,|\,\Lambda}\left(1 - \frac{1}{2^B}\right). \tag{25}$$

Furthermore, for any sequence of signature codebooks $\{\mathcal{V}_N\}$ with associated SINRs $\{\beta_{\mathcal{V}_N}\}$, $\limsup \beta_{\mathcal{V}_N} \leq \beta^{\infty}_{\mathsf{mmse;rvq}}$.

In other words, RVQ achieves the maximum large system SINR, which depends on the distribution of $\beta_{\mathsf{mmse},j}$. The proof is given in Appendix C, and again relies on the asymptotic theory of extreme order statistics [32].

To evaluate the large system SINR for RVQ in Theorem 3, $\beta^{\infty}_{\mathsf{mmse;rvq}}$, we require the distribution of the SINR $\beta_{\mathsf{mmse,j}}$ with an MMSE receiver, given $\boldsymbol{S}_1$. This cdf is given by (56) in Appendix A (for $K > N$), where in this case the λ_i's are the eigenvalues of $\boldsymbol{R}_1^{-1}$. As for the matched-filter receiver, the inverse of the cdf given in (56) appears to be difficult to evaluate analytically, which prevents us from evaluating (25) directly. An upper bound on the large system SINR can be obtained by following the approach used for the matched filter. Namely, we condition on the codebook $\mathcal{V}$, and replace the random variables $\beta_{\mathsf{mmse,j}}$ by independent random variables having the same distribution. This gives

$$\beta^{\infty}_{\mathsf{mmse;rvq}} \leq \lim_{(N,K,B)\to\infty} G^{-1}_{\beta\,|\,\mathcal{V}}\left(1 - \frac{1}{2^B}\right) \tag{26}$$

where $G_{\beta\,|\,\mathcal{V}}(\,\cdot\,)$ is the distribution of $\beta_{\mathsf{mmse,j}}$ conditioned on $\mathcal{V}$. The exact expression for $G_{\beta\,|\,\mathcal{V}}(\,\cdot\,)$ is unknown, and appears to be difficult to evaluate. However, we can resort to a Gaussian approximation for $G_{\beta\,|\,\mathcal{V}}(\,\cdot\,)$, which is motivated by the following lemma.

Lemma 1 [17, Theorem 4.5]: As $(N,K) \to \infty$

$$\sqrt{N}(\beta_{\mathsf{mmse,j}} - \beta^*) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma^2_\beta\right) \tag{27}$$

where β^* is the large system SINR with randomly assigned signatures given in (11) and

$$\sigma^2_\beta = \frac{2\beta^*(1+\beta^*)^2}{\sigma^2_n(1+\beta^*)^2 + \bar{K}} - 2\beta^{*2}. \tag{28}$$

Hence, for large N, $G_{\beta|\mathcal{V}}(\,\cdot\,)$ is approximately Gaussian with mean β^* and variance $\frac{1}{N}\sigma^2_\beta$. Evaluating (26) with this approximation gives[5]

$$\beta^{\infty}_{\mathsf{mmse;rvq}} \approx \tilde{\beta}^{\infty}_{\mathsf{mmse;rvq}} = \beta^* + \sigma_\beta\sqrt{(2\log 2)\bar{B}}. \tag{29}$$

[5]The limit in (26) for fixed K and N is evaluated in [32, Sec. 2.3.2] for a zero-mean, unit-variance Gaussian cdf $G_{\beta|\mathcal{V}}(\,\cdot\,)$. Evaluation of the limit in (25) with a different mean and variance is a straightforward extension.

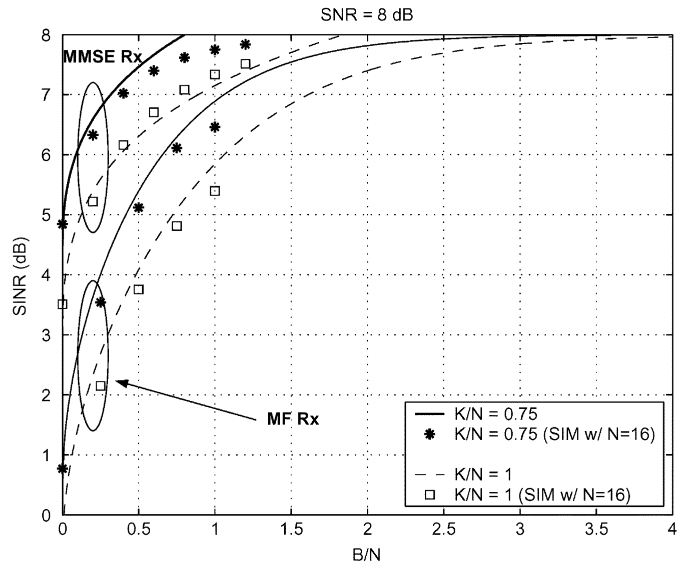

Fig. 1. SINR (in decibels) for RVQ with matched-filter and MMSE receivers versus normalized feedback $\bar{B}$. Both large system and simulated results are shown.

This approximation is accurate for small $\bar{B}$. Namely, when $\bar{B} = 0$, $\beta^{\infty}_{\mathsf{mmse;rvq}} = \tilde{\beta}^{\infty}_{\mathsf{mmse;rvq}} = \beta^*$. However, as $\bar{B} \to \infty$, $\tilde{\beta}^{\infty}_{\mathsf{mmse;rvq}} \to \infty$, and hence, exceeds the single-user bound. This is because we are approximating $G_{\beta\,|\,\mathcal{V}}(\,\cdot\,)$ by a Gaussian pdf, which has infinite support.

C. Numerical Example

Fig. 1 shows plots of large system SINR with RVQ versus normalized feedback $\bar{B}$ for both the matched-filter and MMSE receivers with $\bar{K} = 0.75$ and 1, and background signal-to-noise ratio (SNR) 8 dB. The large system upper bound for the matched filter is shown, which is obtained by solving (19). The MMSE results, given by (29), are based on the Gaussian approximation. Also shown in the figure are discrete points corresponding to Monte Carlo simulations with $N = 16$. The large system results for the matched filter accurately predict finite system performance. These results show that the Gaussian approximation for the MMSE receiver is especially accurate for small $\bar{B}$.

For the example shown, to achieve the single-user bound, the MMSE receiver requires about one bit per dimension, whereas the matched filter requires about two bits per dimension. However, the matched filter shows a much larger performance improvement with a small amount of feedback, relative to no feedback.

IV. Reduced-Rank Signature With Scalar Quantization

A drawback of vector quantization schemes in general is complexity. Namely, the size of the codebook, and hence the complexity, grows exponentially with the number of feedback bits B. We therefore seek simpler, suboptimal quantization schemes. For example, one possibility is to quantize each element of the optimal signature with the same scalar quantizer. To reduce the number of quantized coefficients, we can apply reduced-rank signature optimization, as proposed in [15]. Namely, the optimal signature is projected onto a D-dimensional subspace, where

$D < N$, which is known at the transmitter. The receiver then relays the D scalar-quantized coefficients back to the transmitter.

With unlimited feedback, the performance of reduced-rank signature optimization increases monotonically with D, i.e., larger D implies more degrees of freedom for interference avoidance. Limited feedback introduces the following tradeoff: as the number of dimensions increases, there are more coefficients to quantize, resulting in higher quantization error, but also more dimensions available for interference avoidance. Hence, there is generally an optimal dimension that depends on the load, the number of feedback bits, and (to a lesser degree) the noise level. In what follows, we first present the large system performance of reduced-rank signature optimization with unlimited feedback as a function of the dimension D, and subsequently evaluate the performance with scalar quantization.[6]

A. Reduced-Rank Performance With Unlimited Feedback

A reduced-rank signature is an $N \times 1$ vector, which is constrained to lie in a D-dimensional subspace, $D \leq N$, i.e.,

$$\boldsymbol{s}_k = \boldsymbol{F}_k \boldsymbol{\alpha}_k \tag{30}$$

where $\boldsymbol{F}_k$ is the $N \times D$ matrix of basis vectors, and $\boldsymbol{\alpha}_k$ is the D-vector of combining coefficients, all for user k. Since the transmitter and receiver have no *a priori* knowledge of the interference, we assume that $\boldsymbol{F}_k$ is a random unitary matrix known to the transmitter and receiver, i.e., $\boldsymbol{F}_k^\dagger \boldsymbol{F}_k = \boldsymbol{I}$. We refer to D as the *rank* of the signature. It represents the available degrees of freedom for interference avoidance.

The optimized SINR (i.e., maximized over $\boldsymbol{\alpha}_k$) increases with D. Note that $D = 1$ corresponds to conventional power control (i.e., the signature is scaled by a constant factor), and $D = N$ corresponds to the optimized (full-rank) signature discussed in Section III. Varying D allows a tradeoff between achievable performance and the number of coefficients to be estimated and fed back to the transmitter.

We wish to select $\boldsymbol{\alpha}_k$ to maximize

$$J = \mathsf{SINR} + \varsigma(||\boldsymbol{F}_k \boldsymbol{\alpha}_k||^2 - 1) \tag{31}$$

where ς is a Lagrange multiplier for the power constraint. The solution to this problem with matched-filter and MMSE receivers is presented in [15]. Namely, the optimal $\boldsymbol{\alpha}_1$ with a matched filter is the eigenvector of $\boldsymbol{F}_1^\dagger \boldsymbol{R}_1 \boldsymbol{F}_1$ corresponding to the minimum eigenvalue, which is the interference power. As $(N, K, D) \to \infty$ with fixed normalized rank $\bar{D} = D/K$ and normalized load $\bar{K} = K/N$, the eigenvalue distribution of the matrix $\boldsymbol{F}_1^\dagger \boldsymbol{R}_1 \boldsymbol{F}_1$ converges weakly to a deterministic function, and the minimum eigenvalue converges to a deterministic value. The large system SINR at the output of the matched filter, evaluated in [15], is

$$\beta_{\mathsf{mf;rr}}^{\infty} = \frac{1}{\sigma_n^2 + \bar{K}(1 - \sqrt{\bar{D}})^2} \tag{32}$$

for $0 \leq \bar{D}, \bar{K} \leq 1$. Setting $\bar{D} = 0$ gives the large system SINR with a random signature, and setting $\bar{D} = 1$ gives the single-user SINR, since there are enough degrees of freedom to completely avoid all interferers.

[6]The performance of reduced-rank signature optimization with RVQ could also be analyzed by combining the following results with the previous analysis of RVQ. Here we only consider scalar quantization of the reduced-rank coefficients to simplify the receiver.

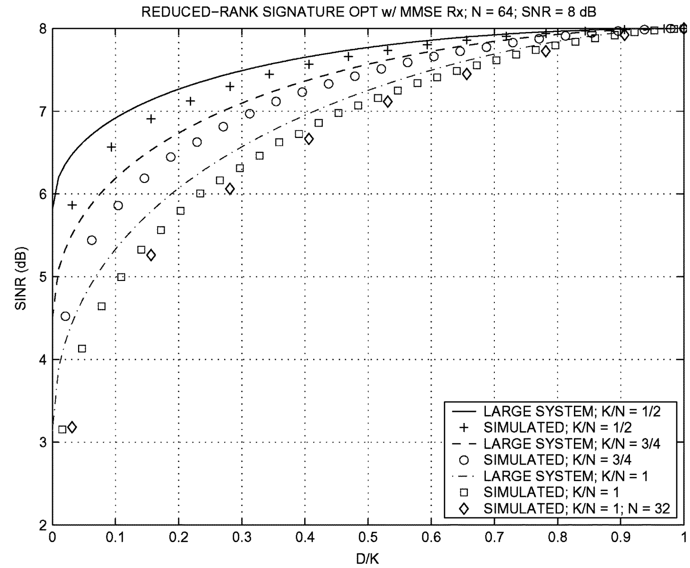

Fig. 2. SINR (in decibels) with the MMSE receiver and reduced-rank signature optimization versus normalized rank $\bar{D}$. Both large system and simulated performance are shown.

For the MMSE receiver, we substitute the expression for SINR from (7) into (31) and maximize with respect to $\boldsymbol{\alpha}_1$. The optimal $\boldsymbol{\alpha}_1$ is the eigenvector of $\boldsymbol{F}_1^\dagger \boldsymbol{R}_1^{-1} \boldsymbol{F}_1$ corresponding to the maximum eigenvalue, which is the corresponding SINR. The large system SINR with reduced-rank signature, $\beta_{\mathsf{mmse;rr}}^{\infty}$, can be evaluated as follows.

Theorem 4: As $(D, K, N) \to \infty$, the SINR with the reduced-rank signature, $\beta_{\mathsf{mmse;rr}}^{\infty}$ converges to the real root of the following quartic equation:

$$b_4 \lambda^4 + b_3 \lambda^3 + b_2 \lambda^2 + b_1 \lambda + b_0 = 0 \tag{33}$$

for $0 \leq \bar{K}, \bar{D} \leq 1$, where b_4 through b_0 can be written as finite polynomials containing $\bar{K}, \bar{D}$, and σ_n^2.

The proof and explicit expressions for b_4 through b_0 are given in Appendix E. There can be multiple real roots of (33); however, only one corresponds to a feasible solution. The proof relies on free probability theory [34]. (A related eigenvalue result for a different matrix is presented in [35].) If $\bar{D} = 0$, then (33) becomes a quadratic equation, the solution to which is the large system SINR with random signatures [29].

Fig. 2 compares the large system SINR with simulation results for $N = 64$ and 32, and the background SNR is 8 dB. Results are shown for loads $\bar{K} = 1/2, 3/4$, and 1. The analytical results are within a small fraction of a decibel from the simulated results. We also note that at $\bar{D} = 0.5$, the SINR for all loads is within 1 dB of the single-user performance.

B. Performance With Scalar Quantization

We now assume that the receiver quantizes each of the D coefficients in the reduced-rank signature vector $\boldsymbol{\alpha}_k$ with the same scalar quantizer, and relays B bits back to the transmitter. As before, we take $k = 1$, and assume random interferers. Each coefficient $[\boldsymbol{\alpha}_1]_i, 1 \leq i \leq D$ is therefore quantized with B/D

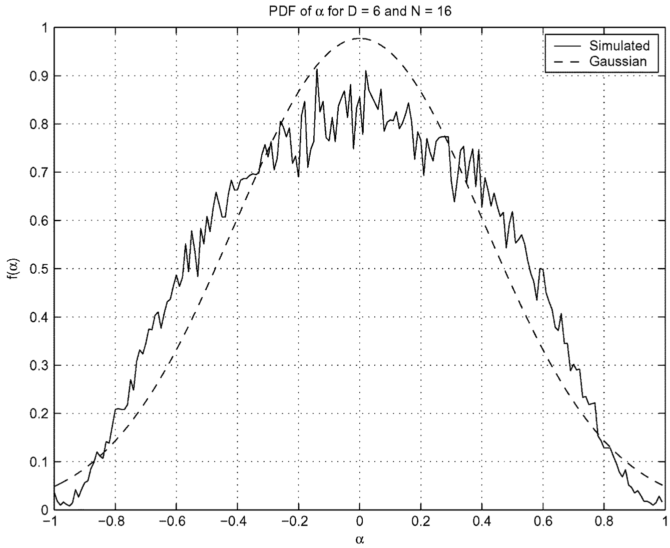

Fig. 3. The empirical pdf of a combining coefficient $[\boldsymbol{\alpha}_1]_i$ for $N = 16$ and $D = 6$, and its Gaussian approximation.

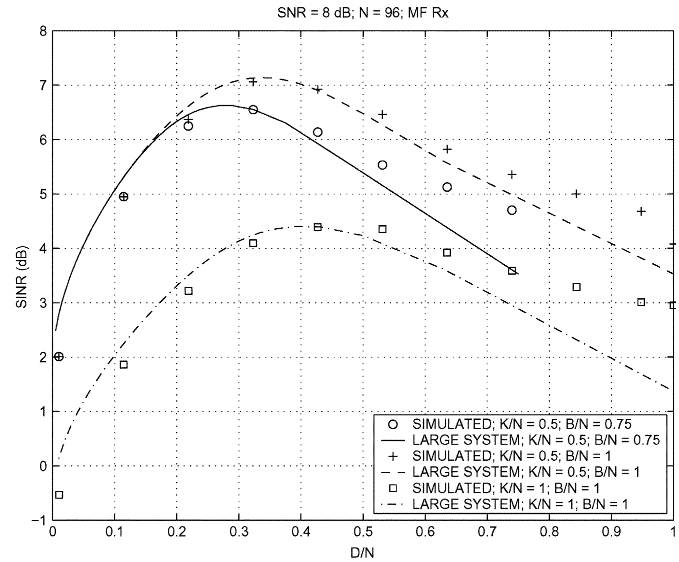

Fig. 4. SINR (decibels) versus normalized rank D/N for the optimized reduced-rank signature with a matched filter. Plots are shown for different loads $\bar{K}$ and normalized feedback $\bar{B}$.

bits. Let $\hat{\boldsymbol{\alpha}}_1$ and $\Delta\boldsymbol{\alpha}_1$ be the quantized coefficient vector and the quantization error vector for user 1, respectively. Hence,

$$\hat{\boldsymbol{\alpha}}_1 = \boldsymbol{\alpha}_1 + \Delta\boldsymbol{\alpha}_1. \tag{34}$$

Here we wish to design a scalar quantizer for each $[\boldsymbol{\alpha}_1]_i$ that minimizes the mean-squared error (MSE) $E(|[\Delta\boldsymbol{\alpha}_1]_i|^2)$. (This is in contrast with a vector quantization scheme, which directly maximizes received SINR.) This is the objective of the Lloyd–Max algorithm [36], [37], which requires the pdf for $[\boldsymbol{\alpha}_1]_i$. This pdf appears to be difficult to obtain exactly; however, numerical observations show that it can be approximated as Gaussian with zero mean and variance $1/D$. An example is shown in Fig. 3.

In what follows, we compute the large system SINR using this quantization scheme for both matched and MMSE filters. In this case, we must let K, N, D, and B all tend to infinity with fixed ratios $\bar{K}, \bar{D}$, and $\bar{B}$.

1) Matched Filter: The quantized $N \times 1$ signature vector for user 1 is

$$\hat{\boldsymbol{s}}_1 = \boldsymbol{F}_1 \hat{\boldsymbol{\alpha}}_1 \tag{35}$$

which must be renormalized to satisfy the power constraint. The corresponding SINR at the output of the matched filter is

$$\beta_{\mathsf{mf;sq}} = \frac{1}{(\hat{\boldsymbol{s}}_1/\|\hat{\boldsymbol{s}}_1\|)^\dagger \boldsymbol{R}_1 (\hat{\boldsymbol{s}}_1/\|\hat{\boldsymbol{s}}_1\|)} = \frac{\hat{\boldsymbol{s}}_1^\dagger \hat{\boldsymbol{s}}_1}{\hat{\boldsymbol{s}}_1^\dagger \boldsymbol{R}_1 \hat{\boldsymbol{s}}_1}. \tag{36}$$

Combining with (34), (35), and (3), and using the fact that $\boldsymbol{F}_1^\dagger \boldsymbol{F}_1 = \boldsymbol{I}$, the averaged SINR is given by

$$\beta_{\mathsf{mf;sq}} = \frac{E[\|\hat{\boldsymbol{\alpha}}_1\|^2]}{\sigma_n^2 E[\|\hat{\boldsymbol{\alpha}}_1\|^2] + E[\hat{\boldsymbol{\alpha}}_1^\dagger \boldsymbol{F}_1^\dagger \boldsymbol{S}_1 \boldsymbol{S}_1^\dagger \boldsymbol{F}_1 \hat{\boldsymbol{\alpha}}_1]} \tag{37}$$

where the expectation is over the distribution of the quantization noise. As $(N, K, D, B) \to \infty$ with fixed $\bar{K}, \bar{D}$, and $\bar{B}$, we show in Appendix F that the SINR in (37) converges to a limit, which can be approximated as

$$\tilde{\beta}^\infty_{\mathsf{mf;sq}} = \frac{1 - q(\bar{D}, \bar{B}, \bar{K})}{\frac{1}{\beta^\infty_{\mathsf{mf;rr}}} + \left(4\sqrt{\bar{D}}\bar{K} - 2\bar{D}\bar{K} - \bar{K} - \sigma_n^2\right) q(\bar{D}, \bar{B}, \bar{K})} \tag{38}$$

where $\beta^\infty_{\mathsf{mf;rr}}$ is defined in (32) and

$$q(\bar{D}, \bar{B}, \bar{K}) = (\pi\sqrt{3})2^{-2\bar{B}/(\bar{D}\bar{K})-1} \tag{39}$$

is the asymptotic quantization noise power defined in [37], and gives an accurate estimate of the actual quantization noise power when the number of feedback bits per reduced-rank coefficient $\bar{B}/(\bar{D}\bar{K})$ is large. (Alternatively, for small $\bar{B}/(\bar{D}\bar{K})$ we can compute $q(\bar{D}, \bar{B}, \bar{K})$ directly from (152) in Appendix F.) The optimal $\bar{D}$, which maximizes $\tilde{\beta}^\infty_{\mathsf{mf;sq}}$, can be easily computed from (38) for given $\bar{B}, \bar{K}$, and SNR.

Fig. 4 compares the large system analytical results obtained from (38) with simulated results $(N = 96)$ for different $\bar{B}$ and $\bar{K}$ and SNR $= 8$ dB. The SINR for user 1 is plotted versus D/N. These results illustrate the tradeoff between the quantization error and the available degrees of freedom D to avoid interference. That is, for small D/N, the performance is limited by the available degrees of freedom, whereas for large D/N, the performance is limited by quantization error. For $\bar{K} = 0.5$ and $\bar{B} = 1$, the maximum SINR is achieved at $D/N = 0.35$, and the difference in SINRs for full-rank and optimized reduced-rank signatures is about 3 dB.

The large system SINR is within a fraction of a decibel from the simulated results for $B/D > 1$. When $B/D = 1$, the approximation (38) becomes inaccurate for large D/N (i.e., corresponding to near-full-rank performance), so that the large

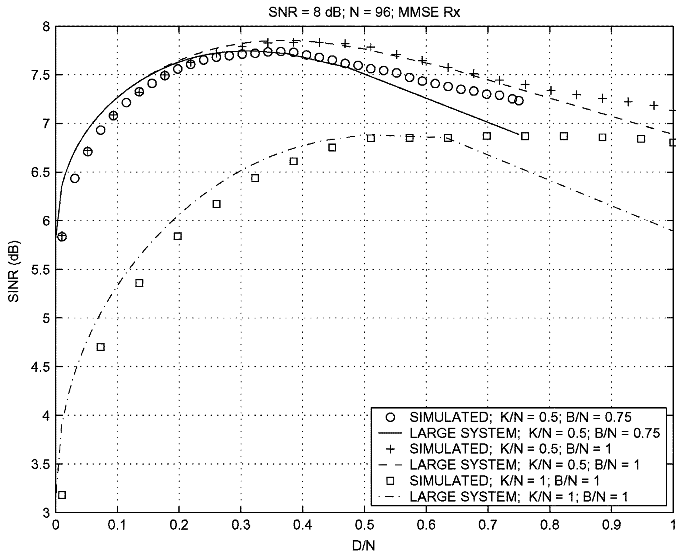

Fig. 5. SINR (decibels) versus normalized rank D/N for the optimized reduced-rank signature with the MMSE receiver. Plots are shown for different loads $\bar{K}$ and normalized feedback $\bar{B}$.

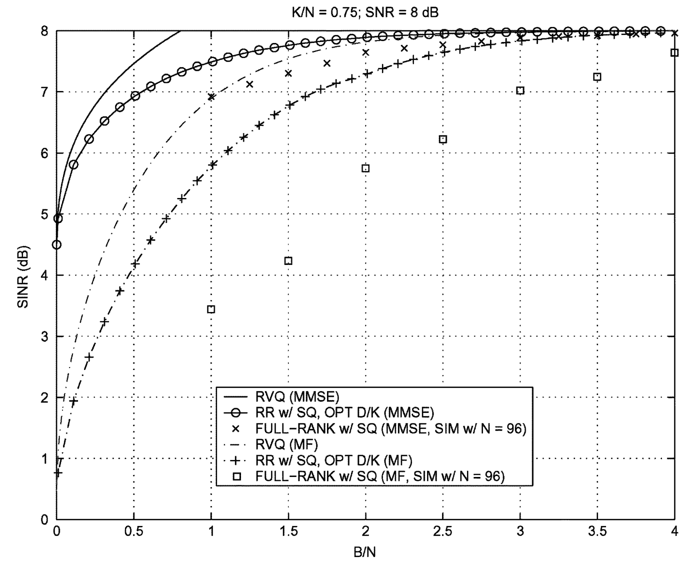

Fig. 6. SINR versus normalized feedback $\bar{B}$ for the optimized reduced-rank signature (RR) with scalar quantization (SQ), the optimized full-rank signature with scalar quantization, and RVQ.

system results deviate more from the simulated results in this region.

2) MMSE Receiver: The SINR with quantized reduced-rank signature and MMSE receiver is given by

$$\beta_{\text{mmse;sq}} = \frac{\hat{\boldsymbol{\alpha}}_1^\dagger \boldsymbol{F}_1^\dagger \boldsymbol{R}_1^{-1} \boldsymbol{F}_1 \hat{\boldsymbol{\alpha}}_1}{\|\hat{\boldsymbol{\alpha}}_1\|^2} \tag{40}$$

where the quantized signature must be renormalized to satisfy the power constraint. In analogy with the matched filter, we can approximate the large system SINR at the output of MMSE receiver as

$$\tilde{\beta}^{\infty}_{\text{mmse;sq}} = \beta^{\infty}_{\text{mmse;rr}} - \frac{\left(\beta^{\infty}_{\text{mmse;rr}} - \beta^*\right) q(\bar{D}, \bar{B}, \bar{K})}{1 - q(\bar{D}, \bar{B}, \bar{K})} \tag{41}$$

where β^* is given in (11) and $\beta^{\infty}_{\text{mmse;rr}}$ is the real root of (33). The derivation of (41) is similar to the derivation of (38), and is omitted. The second term in (41) represents the performance degradation due to quantization.

Fig. 5 compares the large system analytical results obtained from (41) with simulated results ($N = 96$) for different $\bar{B}$ and $\bar{K}$ with SNR $= 8$ dB. The SINR for user 1 is again shown versus D/N. As expected, for given D/N the SINR with the MMSE receiver is higher than that with the matched filter. For the larger load, $\bar{K} = 1$, the tradeoff between quantization error and degrees of freedom for interference avoidance is not apparent, since the quantized full-rank signature with one bit per coefficient performs nearly as well as a quantized reduced-rank signature. This is in contrast to prior results with the matched filter, for which the preceding tradeoff leads to a more pronounced maximum. This can be attributed to the additional interference suppression capability of the MMSE receiver. The large system results accurately predict the simulated results except when B/D is close to one.

Fig. 6 compares the large system performance of the scalar-quantized reduced-rank signature, the scalar-quantized full-rank signature, and RVQ. The figure shows SINR versus normalized feedback $\bar{B}$ for both the MMSE and matched-filter receivers. The reduced-rank results correspond to the optimal $\bar{D}$. The large system SINRs for RVQ are evaluated from Theorem 1 for the matched filter and (29) for the MMSE receiver. The reduced-rank SINRs are evaluated from (38) and (41). The full-rank results are from simulation with $N = 96$. These results show that the SINR for the reduced-rank scheme with optimal rank is within 1 dB of RVQ, and is substantially greater than the SINR for full-rank optimization with scalar quantization. In particular, scalar quantization of the full-rank signature is not possible for $\bar{B} < 1$. The gain from using optimized reduced-rank signatures, relative to full-rank signatures, is greater for the matched filter than for the MMSE receiver, although the MMSE receiver gives significantly better performance than the matched filter. We remark that the reduced-rank results are insensitive to a suboptimal choice of $\bar{D}$, as long as it is in the vicinity of the optimal $\bar{D}$.

V. Group Signature Optimization With Limited Feedback

So far, we have considered the situation in which only one user optimizes his signature in the presence of fixed, random interfering signatures. Now we consider the situation in which a *group* of users optimizes their signatures in the presence of fixed, random interference. The objective is to maximize the average SINR over all users in the group. For example, the optimizing users could be mobiles within a particular cell, in which case the interfering users are in other cells. The receiver can select a set of signatures for the group of users from an expanded codebook, and feed back the corresponding index bits. We again analyze the large system performance of RVQ with matched and MMSE filters, and compare with scalar-quantized reduced-rank signatures.

Let K_a be the number of optimized signatures and K_b be the number of interfering signatures. The total number of users is $K = K_a + K_b$. Suppose that $\{\boldsymbol{s}_k, 1 \le k \le K_a\}$ is the signature set for the optimizing users and $\{\boldsymbol{p}_i, 1 \le i \le K_b\}$ is the signature set for the interfering users. The SINR at the output of the matched filter averaged over the K_a users is given by

$$\beta_{\mathsf{mf;gr}} = \frac{K_a}{K_a\sigma_n^2 + \sum_{k=1}^{K_a}\sum_{l\neq k}(\boldsymbol{s}_k^\dagger \boldsymbol{s}_l)^2 + \sum_{k=1}^{K_a}\sum_{i=1}^{K_b}(\boldsymbol{s}_k^\dagger \boldsymbol{p}_i)^2} \tag{42}$$

$$= \frac{1}{\sigma_n^2 + \frac{1}{K_a}\mathrm{tr}\{\boldsymbol{S}^\dagger \boldsymbol{S}\boldsymbol{S}^\dagger \boldsymbol{S}\} - 1 + \frac{1}{K_a}\mathrm{tr}\{\boldsymbol{P}^\dagger \boldsymbol{S}\boldsymbol{S}^\dagger \boldsymbol{P}\}} \tag{43}$$

where $\boldsymbol{S} = [\boldsymbol{s}_1\ \boldsymbol{s}_2\ \cdots\ \boldsymbol{s}_{K_a}]$ and $\boldsymbol{P} = [\boldsymbol{p}_1\ \boldsymbol{p}_2\ \cdots\ \boldsymbol{p}_{K_b}]$. Given B feedback bits, the set of signatures can be selected from a codebook $\mathcal{V} = \{\boldsymbol{V}_j, 1 \le j \le 2^B\}$, namely, the receiver selects

$$\boldsymbol{S} = \arg\max_j \beta_{\mathsf{mf;gr}}(\boldsymbol{V}_j). \tag{44}$$

The optimal signature matrix $\boldsymbol{S}$ with unlimited feedback has columns, which are eigenvectors of $\boldsymbol{P}\boldsymbol{P}^\dagger$. Here we define the RVQ codebook as having elements, which are independent and isotropically distributed ($\boldsymbol{V}^\dagger \boldsymbol{V} = \boldsymbol{I}$). The denominator of (43) therefore simplifies to $\sigma_n^2 + \frac{1}{K_a}\mathrm{tr}\{\boldsymbol{P}^\dagger \boldsymbol{S}\boldsymbol{S}^\dagger \boldsymbol{P}\}$, and the minimum interference is

$$W_g = \min_j \left\{ Y_j = \frac{1}{K_a}\mathrm{tr}\{\boldsymbol{P}^\dagger \boldsymbol{V}_j \boldsymbol{V}_j^\dagger \boldsymbol{P}\}\right\} \tag{45}$$

which is a random variable whose distribution is difficult to evaluate for finite K and N. We again study the asymptotic, or large system interference as $(N, K_a, K_b, B) \to \infty$ with fixed ratios $\bar{K}_a = K_a/N$, $\bar{K}_b = K_b/N$, and $\hat{B} = B/(K_a N)$. Because there are now $K_a N$ signature coefficients, and K_a grows linearly with N, the number of feedback bits increases as N^2.

In analogy with Theorem 1, W_g converges in the mean-square sense to

$$W_g^\infty = \lim_{(N,K_a,K_b,B)\to\infty} F_{Y\,|\,\boldsymbol{P}}^{-1}\left(\frac{1}{2^B}\right) \tag{46}$$

where $F_{Y\,|\,\boldsymbol{P}}(\,\cdot\,)$ is the cdf for Y_j given the matrix $\boldsymbol{P}$. Evaluation of $F_{Y\,|\,\boldsymbol{P}}(\,\cdot\,)$ remains an open problem. Hence, as in the single-user analysis, to obtain a lower bound on large system interference power, we condition on the codebook $\mathcal{V}$ with random $\boldsymbol{P}$. Given $\boldsymbol{V}_j$, the central limit theorem implies that each diagonal element of $\boldsymbol{P}^\dagger \boldsymbol{V}_j$ converges to a Gaussian random variable as $N \to \infty$, so that Y_j converges to a Gamma random variable with mean K_b/N and variance $2K_b/(K_a N^2)$. This cdf, which we denote as $F_{Y\,|\,\mathcal{V}}(\,\cdot\,)$, is similar to the cdf for the interference power in the single-user case discussed in Section III, except that here the variance goes to zero as $1/N^2$ instead of $1/N$. In analogy with the single-user analysis, it follows that

$$W_g^\infty \ge \bar{W}_g^\infty = \lim_{(N,K_a,K_b,B)\to\infty} F_{Y\,|\,\mathcal{V}}^{-1}\left(\frac{1}{2^B}\right)$$

where $\bar{W}_g^\infty$ satisfies

$$(\bar{W}_g^\infty)^{\bar{K}_b} \mathrm{e}^{-\bar{W}_g^\infty} = \left(\frac{\bar{K}_b}{\mathrm{e}}\right)^{\bar{K}_b} 2^{-2\hat{B}} \tag{47}$$

for $\bar{K} \ge 0$ and $\hat{B} \ge 0$. For $\bar{K} \le 1$, the bound is again tight when $\hat{B} = 0$, in which case $\bar{W}_g^\infty = \bar{K}_b$, and when $\hat{B} \to \infty$, in which case $\bar{W}_g^\infty = 0$. The numerical example, which follows, shows that the bound is very close to the simulated results. The large system SINR is upper-bounded by

$$\beta_{\mathsf{mf;rvq\text{-}gr}}^\infty = \frac{1}{\sigma_n^2 + W_g^\infty} \le \frac{1}{\sigma_n^2 + \bar{W}_g^\infty}. \tag{48}$$

For the MMSE receiver with group signature optimization the performance objective is sum MSE given by

$$\mathcal{M} = 1 - \frac{1}{K_a}\sum_{k=1}^{K_a} \boldsymbol{s}_k^\dagger \boldsymbol{R}^{-1} \boldsymbol{s}_k \tag{49}$$

$$= 1 - \frac{1}{K_a}\mathrm{tr}\{\boldsymbol{S}^\dagger \boldsymbol{R}^{-1} \boldsymbol{S}\} \tag{50}$$

where the received covariance matrix $\boldsymbol{R} = \boldsymbol{S}\boldsymbol{S}^\dagger + \boldsymbol{P}\boldsymbol{P}^\dagger + \sigma_n^2 \boldsymbol{I}$. To evaluate the limiting performance, we require the cdf of $\mathcal{M}$ for finite N. This cdf is difficult to determine exactly, so that we instead minimize the following upper bound on the average MSE derived in Appendix G:

$$\mathcal{M} \le 1 - \frac{1}{1 + \sigma_n^2 + \frac{1}{K_a}\mathrm{tr}\{\boldsymbol{P}^\dagger \boldsymbol{S}\boldsymbol{S}^\dagger \boldsymbol{P}\}}. \tag{51}$$

Minimizing this bound is equivalent to minimizing W_g. Therefore, the large system average MSE satisfies

$$\lim_{(N,K_a,K_b,B)\to\infty} \mathcal{M} = \mathcal{M}^\infty \le 1 - \frac{1}{1 + \sigma_n^2 + W_g^\infty} \tag{52}$$

where convergence is in the mean-square sense, W_g^∞ is accurately approximated by $\bar{W}_g^\infty$, which satisfies (47), and the latter limit converges for $\bar{K} \ge 0$ and $\hat{B} \ge 0$.

To simplify the feedback scheme, we again apply reduced-rank signature optimization with a scalar quantizer. For group optimization, each signature $\boldsymbol{s}_k$ is projected onto a different random subspace spanned by the columns of the matrix $\boldsymbol{F}_k$ [15]. Here we assume that $\{\boldsymbol{F}_k, 1 \le k \le K_a\}$ is a set of independent, random unitary matrices known *a priori* to both the transmitter and receiver.

The optimized set of reduced-rank signatures $\{\boldsymbol{\alpha}_k, 1 \le k \le K_a\}$ corresponds to a fixed point in which each $\boldsymbol{\alpha}_k$ is an eigenvector of the the corresponding interference-plus-noise covariance matrix (i.e., $\boldsymbol{F}_k^\dagger \boldsymbol{R}_k \boldsymbol{F}_k$ for the matched filter and $\boldsymbol{F}_k^\dagger \boldsymbol{R}_k^{-1} \boldsymbol{F}_k$ for the MMSE filter). Such a fixed point can typically be found numerically by alternately updating the users' signatures and receivers until the performance objective (i.e., average SINR

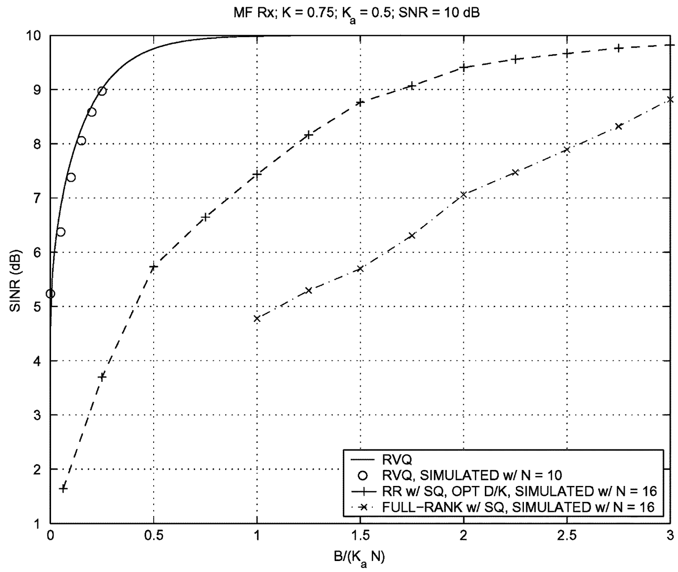

Fig. 7. SINR (decibels) versus normalized feedback $\hat{B}$ with group signature optimization and a matched filter. Results are shown for RVQ, and scalar-quantized reduced-rank signatures.

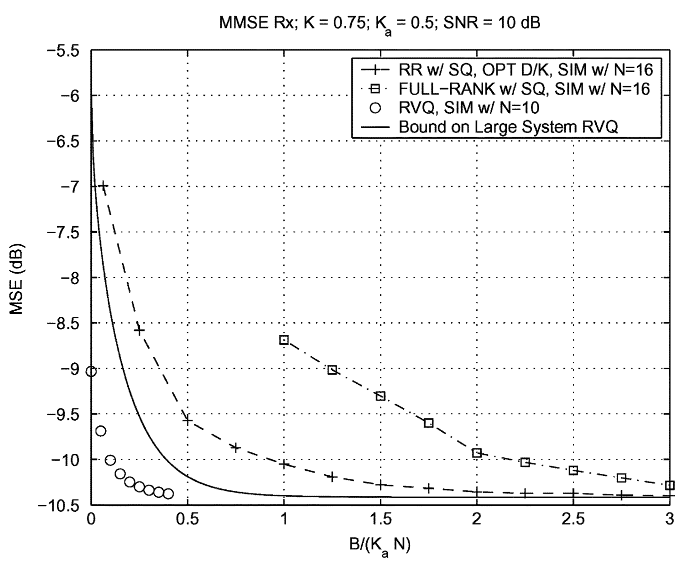

Fig. 8. Average MSE (decibels) versus normalized feedback $\hat{B}$ with group signature optimization and an MMSE receiver. Results are shown for RVQ and scalar-quantized reduced-rank signatures.

or sum MMSE) converges.[7] Determination of the large system performance of group reduced-rank signature optimization is an open problem. For the numerical results, which follow, the fixed points are computed numerically, and as in Section IV, each element of each signature $\boldsymbol{\alpha}_k, k = 1, \ldots, K$, is quantized with a scalar quantizer optimized for a Gaussian pdf with $B/(DK_a)$ feedback bits.

Figs. 7 and 8 compare the performance of group RVQ and scalar-quantized reduced-rank signatures with matched-filter and MMSE receivers. Fig. 7 for the matched filter shows SINR averaged across users, and Fig. 8 for the MMSE receiver shows averaged MSE. Parameters are $\bar{K}_a = 0.5, \bar{K}_b = 0.25$, and SNR $= 10$ dB. Simulated RVQ performance with $N = 10$ is also shown, along with simulated results for scalar-quantized reduced- and full-rank (i.e., $D = N$) signatures with $N = 16$. The reduced-rank results are optimized with respect to the rank D. (Note that the size of the RVQ codebook grows as $2^{\bar{K}_a \hat{B} N^2}$, so that the RVQ simulation is restricted to relatively small N.)

For the matched-filter receiver, the large system upper bound on RVQ performance accurately estimates the simulated performance. For the MMSE receiver, the approximate large system performance is shown. The results show that RVQ achieves the single-user bound with $\hat{B} < 1$. Reduced-rank performance shows a significant degradation relative to RVQ (e.g., 2.5 dB with the matched filter when $\hat{B} = 1$), but is still significantly better than direct scalar quantization of the optimal (full-rank) signatures (again, about 2.5 dB for the previous example). The MMSE results show that to achieve a target MSE, scalar-quantized optimal signatures require about one bit per coefficient more feedback than scalar-quantized reduced-rank signatures.

VI. Conclusion

We have analyzed the performance of CDMA signature optimization with linear receivers and limited feedback. A key feature of this analysis is the computation of large system performance, where in addition to keeping the ratio of users to processing gain fixed, the number of feedback bits per signature coefficient is also constant. In this large system limit, the performance of RVQ gives an upper bound on the performance of any signature quantization scheme. For single-user signature optimization the numerical example shows that one feedback bit per signature coefficient with a matched-filter receiver gives a received SINR, which is about 1 dB from the single-user bound. With an MMSE receiver, one feedback bit per coefficient can essentially achieve the single-user SINR. Group optimization requires significantly less feedback (0.25–0.5 bit per coefficient) to achieve this performance.

To simplify the signature quantizer at the receiver, we considered reduced-rank signature optimization with a scalar quantizer. Selecting the optimal rank gives a substantial increase in SINR over (full-rank) scalar quantization of the optimal signature, especially with the matched-filter receiver. Our numerical results for single-user optimization show that the reduced-rank scheme with optimal rank performs in the vicinity of the RVQ bound. Specifically, to achieve an SINR within 0.5 dB of the single-user bound with an MMSE receiver, reduced-rank optimization requires about 0.5 bit per coefficient more than RVQ. For group adaptation, the reduced-rank results are further away from this bound, but still offer substantial gains relative to scalar quantization of the optimal signatures.

Although we have considered an idealized CDMA model, some of our results have wider applicability. For example, the asymptotic optimality of RVQ is essentially a consequence of isotropic interfering signatures, hence, we expect that RVQ is asymptotically optimal in more general scenarios than the one considered here. For example, those scenarios include i.i.d. power assignments across users, asynchronous users with independent, uniformly distributed delays, flat fading with

[7]The signatures can also be updated sequentially across users, as in [11]. Here we do not investigate associated convergence and uniqueness issues, but note that for the numerical results, which follow, the iterative algorithms simulated consistently converged to the same fixed point from different initial conditions.

independent channel gains, and any frequency-selective fading model in which the set of received signatures are isotropically distributed. However, computing the large system performance in those scenarios may be difficult, since the interference distribution is more complicated than for the idealized CDMA model.

The techniques discussed here can also be applied to multiantenna systems, where limited feedback is used to determine spatial (or possibly space–time) signatures [26], [27]. Here (and in [26], [27]) we have assumed slowly changing interference and channels so that the receiver has sufficient time to obtain an accurate estimate of the optimal signature, and feed back the quantized coefficients to the transmitter. Additional issues for future study include the effect of signature estimation at the receiver with limited training, and the performance in the presence of channel mismatch due to time-selective fading. Finally, interesting systems issues include throughput optimization over a full-duplex link, taking into account the feedback overhead (see [38]), and evaluating feedback strategies in the context of other network configurations (e.g., the downlink).

APPENDIX

A. Proof of Theorem 1

Let the number of vectors in codebook $\mathcal{V}$ be $n = 2^B$. As $n \to \infty$, the cdf of $W = \min_j\{X_j\}$ for a given $\boldsymbol{S}_1$, denoted as $F_{W\,|\,\boldsymbol{S}_1}$, becomes degenerate at the minimum eigenvalue of $\boldsymbol{S}_1\boldsymbol{S}_1^\dagger$, which is the minimum possible value of X_j. The following lemma states that with appropriate normalizing constants c_n and d_n, $(W - c_n)/d_n$ converges weakly to a Weibull random variable L with cdf F_L.

Lemma 2: There exist c_n and d_n such that as $n \to \infty$

$$F_{W\,|\,\boldsymbol{S}_1}(c_n + d_n x) \longrightarrow F_L(x) \tag{53}$$

for all x where

$$F_L(x) = \begin{cases} 1 - \mathrm{e}^{-x^{-\tau}}, & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{54}$$

and $\tau = N - 1$.

Proof: This follows from [32, Theorem 2.1.5]. First we observe that with RVQ, the interference terms X_j's for given $\boldsymbol{S}_1$ are i.i.d. since the $\boldsymbol{v}_j$'s are i.i.d. According to [32, Theorem 2.1.5], the lemma holds provided that

$$\lim_{t\to-\infty} \frac{F_{X|\boldsymbol{S}_1}(\alpha_F - 1/tx)}{F_{X|\boldsymbol{S}_1}(\alpha_F - 1/t)} = x^{-\tau}, \qquad \tau > 0. \tag{55}$$

where $\alpha_F = \inf\{x : F_{X|\boldsymbol{S}_1}(x) > 0\}$. With isotropically distributed $\boldsymbol{v}_j$'s, the cdf for X_j conditioned on $\boldsymbol{S}_1$, assuming $K \geq N$, is given by [30]

$$F_{X\,|\,\boldsymbol{S}_1}(x) = \sum_{p=s+1}^{N} \frac{(x-\lambda_p)^{N-1}}{\Pi_{l=1,l\neq p}^{N}(\lambda_l - \lambda_p)}, \quad \text{for } \lambda_{s+1} \leq x \leq \lambda_s,\ s = 1,\ldots,N-1 \tag{56}$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ are the ordered eigenvalues of $\boldsymbol{S}_1\boldsymbol{S}_1^\dagger$. The corresponding pdf for an underloaded system $(K \leq N)$ is somewhat more complicated, so that for convenience, in what follows we only consider an overloaded system $(K \geq N)$. An alternative approach to proving the theorem for an underloaded system, which does not rely on a specific form for the distribution $F_{X\,|\,\boldsymbol{S}_1}$, is given in Appendix C (Proof of Theorem 3).

For small x such that $\lambda_N \leq x \leq \lambda_{N-1}$

$$F_{X\,|\,\boldsymbol{S}_1}(x) = (x-\lambda_N)^{N-1}/\Pi_{l=1}^{N-1}(\lambda_l - \lambda_N).$$

Substituting $\alpha_F = \lambda_N$ and $F_{X\,|\,\boldsymbol{S}_1}(x)$ with small x into (55), we can evaluate the limit as follows:

$$\begin{aligned} &\lim_{t\to-\infty} \frac{F_{X\,|\,\boldsymbol{S}_1}(\lambda_N - 1/tx)}{F_{X\,|\,\boldsymbol{S}_1}(\lambda_N - 1/t)} \\ &\quad = \lim_{t\to-\infty} \frac{((\lambda_N - 1/tx) - \lambda_N)^{N-1}}{((\lambda_N - 1/t) - \lambda_N)^{N-1}} && (57) \\ &\quad = x^{-(N-1)}. && (58) \end{aligned}$$

Since $N - 1 > 0$ for $N > 1$, we can apply [32, Theorem 2.1.5]. □

Theorem 2.1.5 in [32] states that the normalizing sequences can be chosen as

$$c_n = \lambda_N \quad \text{and} \quad d_n = F_{X\,|\,\boldsymbol{S}_1}^{-1}\left(\frac{1}{n}\right) - \lambda_N. \tag{59}$$

From Lemma 2 and [39, Theorem 25.1.2]

$$E\left[\frac{W - c_n}{d_n}\right] \to E[L] \tag{60}$$

as $n \to \infty$. The sequence $\{d_n\}$ is deterministic and $E[L] = \Gamma[1 + (N-1)^{-1}]$ [40]. Taking the large system limit gives

$$\begin{aligned} &\lim_{(N,K,n)\to\infty} E[W] \\ &\quad = \lim_{(N,K,n)\to\infty} c_n + d_n\Gamma\left(1 + \frac{1}{N-1}\right) && (61) \\ &\quad = \lim_{(N,K,n)\to\infty} c_n + d_n && (62) \\ &\quad = \lim_{(N,K,n)\to\infty} F_{X\,|\,\boldsymbol{S}_1}^{-1}\left(\frac{1}{n}\right) && (63) \end{aligned}$$

where we have used the fact that $\Gamma(1) = 1$. Furthermore, as $(N, K, n) \to \infty$, the variance

$$E[W^2] - (E[W])^2 \to d_n^2 E[L^2] - d_n^2 = 0 \tag{64}$$

since $E[L^2] = 1$. This establishes that W converges to

$$W^\infty = \lim_{(N,K,B)\to\infty} F_{X\,|\,\boldsymbol{S}_1}^{-1}\left(\frac{1}{2^B}\right) \tag{65}$$

in the mean-square sense. Since the inverse of the cdf $F_{X\,|\,\boldsymbol{S}_1}$ is difficult to analyze, we are not able to evaluate the limit in (65) directly. Instead, we derive a lower bound for this limit.

Given $\boldsymbol{v}_j$, and assuming $\boldsymbol{S}_1$ is random with i.i.d. Gaussian elements each with zero mean and variance $1/N$, the interference power X_j is a Gamma random variable with pdf given by (21).

In this case, the X_j's given $\mathcal{V}$ are *not* independent. It is shown in the Proof of Theorem 2 (see Appendix B) that replacing the X_j's with independent random variables having the same distribution gives a lower bound on the expected minimum interference power. Specifically, this lower bound is

$$\lim_{(N,K,B)\to\infty} E_{\boldsymbol{S}_1}[\min\{X_1,\ldots,X_{2^B}\}\,|\,\mathcal{V}] \geq \lim_{(N,K,B)\to\infty} F^{-1}_{X|\mathcal{V}}\left(\frac{1}{2^B}\right). \quad (66)$$

Next we evaluate the right-hand side of (66), which gives the lower bound in the theorem.

From (21), we wish to find the limit as $(N,K,B)\to\infty$ of the w that satisfies

$$F_{X\,|\,\mathcal{V}}(w) = \frac{(N/2)^{K/2}}{\Gamma(K/2)}\int_0^w x^{K/2-1}\mathrm{e}^{-Nx/2}\,\mathrm{d}x \quad (67)$$

$$= \frac{1}{2^B}. \quad (68)$$

We first rewrite this as

$$\left[\int_0^w x^{K/2-1}\mathrm{e}^{-Nx/2}\,\mathrm{d}x\right]^{\frac{2}{N}} = \left[\frac{\Gamma(K/2)}{(N/2)^{K/2}2^B}\right]^{\frac{2}{N}}. \quad (69)$$

In the large system limit, the right-hand side of (69) converges to

$$\lim_{(N,K,B)\to\infty}\left[\frac{\Gamma(K/2)}{(N/2)^{\frac{1}{2}K}2^B}\right]^{\frac{2}{N}}$$

$$= \lim_{(N,K,B)\to\infty}\left[\frac{\mathrm{e}^{-\frac{1}{2}K}(K/2)^{(\frac{1}{2}K-\frac{1}{2})}\sqrt{2\pi}(1+O(1/K))}{(N/2)^{\frac{1}{2}K}2^B}\right]^{\frac{2}{N}} \quad (70)$$

$$= \bar{K}^{\bar{K}}\mathrm{e}^{-\bar{K}}2^{-2\bar{B}}\lim_{(N,K,B)\to\infty}\left[2\sqrt{\frac{\pi}{K}}\left(1+O\left(\frac{1}{K}\right)\right)\right]^{\frac{2}{N}} \quad (71)$$

$$= \left(\frac{\bar{K}}{\mathrm{e}}\right)^{\bar{K}}2^{-2\bar{B}} \quad (72)$$

where we have applied Stirling's formula for $\Gamma(K/2)$. We will show that the limit of the left-hand side of (69) is upper- and lower-bounded by $(\bar{W}^\infty)^{\bar{K}}\mathrm{e}^{-\bar{W}^\infty}$, which will complete the proof.

Let $h(x) = x^{K/2-1}\mathrm{e}^{-Nx/2}$, which is the integrand in (69). To derive bounds on the left-hand side of (69), we first show that for large enough N and K, $h(x)$ is convex for $x\in(0,w)$. A direct calculation shows that $h''(x)>0$ for

$$x\in\left(0,\bar{K}-\frac{2}{N}-\frac{\sqrt{2K-2}}{N}\right)$$

which implies that $h(x)$ is convex in that region. As $(N,K,B)\to\infty, w\to\bar{W}^\infty<\bar{K}$ for $\bar{B}>0$. Hence, we can always choose N and K large enough so that

$$w\in\left(0,\bar{K}-\frac{2}{N}-\frac{\sqrt{2K-2}}{N}\right).$$

Since for large enough K and N, $h(x)$ is convex in the region of the integral

$$\int_0^w h(x)\,\mathrm{d}x \leq \frac{1}{2}h(w)w \quad (73)$$

$$= \frac{1}{2}w^{K/2}\mathrm{e}^{-Nw/2} \quad (74)$$

so that

$$\lim_{\substack{(N,K,B)\to\infty\\ w\to\bar{W}^\infty}}\left[\int_0^w h(x)\,\mathrm{d}x\right]^{\frac{2}{N}} \leq \lim_{\substack{(N,K,B)\to\infty\\ w\to\bar{W}^\infty}}\left(\frac{1}{2}\right)^{\frac{2}{N}} w^{K/N}\mathrm{e}^{-w} \quad (75)$$

$$= (\bar{W}^\infty)^{\bar{K}}\mathrm{e}^{-\bar{W}^\infty}. \quad (76)$$

The area under $h(x)$ for $x\in(0,w)$ is lower-bounded by the area of the triangle formed by the set of points $\{(w,0),(w,h(w)),(w-h(w)/h'(w),0)\}$. (The last point is the x-intercept of the line tangent to $h(x)$ at $x=w$.) Hence, we have

$$\int_0^w h(x)\,\mathrm{d}x \geq \frac{1}{2}\frac{[h(w)]^2}{h'(w)} \quad (77)$$

$$= \frac{1}{(K-2)\left(1-\frac{w}{K-2/N}\right)}w^{K/2}\mathrm{e}^{-Nw/2} \quad (78)$$

and

$$\lim_{\substack{(N,K,B)\to\infty\\ w\to\bar{W}^\infty}}\left[\int_0^w h(x)\,\mathrm{d}x\right]^{\frac{2}{N}} \geq \lim_{\substack{(N,K,B)\to\infty\\ w\to\bar{W}^\infty}}\left[\frac{1}{(K-2)\left(1-\frac{w}{\bar{K}-2/N}\right)}\right]^{\frac{2}{N}} w^{K/N}\mathrm{e}^{-w} \quad (79)$$

$$= (\bar{W}^\infty)^{\bar{K}}\mathrm{e}^{-\bar{W}^\infty}. \quad (80)$$

This completes the proof.

B. Proof of Theorem 2

Let $I_{j;N} = \frac{1}{N}\sum_{k=1}^K|\boldsymbol{v}_j^\dagger\boldsymbol{s}_k|^2$ denote the interference power associated with the jth signature $\boldsymbol{v}_j\in\mathcal{V}_N$ for a given $\boldsymbol{S}_1$, and let $F_{I;N}$ denote the cdf for $I_{j;N}$ conditioned on $\boldsymbol{S}_1$. Note that because the interfering signatures are i.i.d., $F_{I;N}$ is invariant to any choice of $\boldsymbol{v}_j$ on the unit sphere, and hence does not depend on j. To prove the theorem we first show that

$$\lim_{(K,N,B)\to\infty} E[\min_{1\leq j\leq 2^B} I_{j;N}] \geq \lim_{(K,N,B)\to\infty} F^{-1}_{I;N}\left(\frac{1}{2^B}\right). \quad (81)$$

The expression on the right-hand side corresponds to a random, independent choice of signatures in $\mathcal{V}_N$ for each N. We subsequently show that the left-hand side is maximized when the $\boldsymbol{v}_j$'s are isotropic, corresponding to RVQ.

To show (81), we observe that because $I_{j;N} \geq 0$, the following statement holds for $\forall c \in \mathbb{R}$:

$$c - \sum_{j=1}^{2^B} (c - I_{j;N})^+ \leq \min_{1 \leq j \leq 2^B} I_{j;N} \tag{82}$$

where

$$(c - I_{j;N})^+ = \begin{cases} c - I_{j;N}, & I_{j;N} \leq c \\ 0, & I_{j;N} > c. \end{cases} \tag{83}$$

Taking the expectation of both sides in (82) and the large system limit, we obtain

$$\lim_{(K,N,B)\to\infty} c - \sum_{j=1}^{2^B} E[(c - I_{j;N})^+] \leq \lim_{(K,N,B)\to\infty} E[\min_{1 \leq j \leq 2^B} I_{j;N}]. \tag{84}$$

The term on the left-hand side within the limit is

$$h(c) = c - 2^B E[(c - I_{j;N})^+] \tag{85}$$

$$= c - 2^B \int_0^c (c - x) f_{I;N}(x)\, \mathrm{d}x \tag{86}$$

$$= c - 2^B c F_{I;N}(c) + 2^B \int_0^c x f_{I;N}(x)\, \mathrm{d}x \tag{87}$$

where $f_{I;N}$ is the pdf for $I_{j;N}$. To tighten the bound, we maximize $h(c)$ with respect to c. Setting

$$\frac{\mathrm{d}h(c)}{\mathrm{d}c} = 1 - 2^B F_{I;N}(c) = 0 \tag{88}$$

gives

$$c^* = F_{I;N}^{-1}\left(\frac{1}{2^B}\right). \tag{89}$$

Substituting for c in (87) and taking the large system limit gives

$$\lim_{(N,K,B)\to\infty} h(c^*) = \lim_{(N,K,B)\to\infty} 2^B \int_0^{c^*} x f_{I;N}(x)\, \mathrm{d}x \tag{90}$$

$$\leq \lim_{(N,K,B)\to\infty} E\left[\min_{1 \leq j \leq 2^B} I_{j;N}\right]. \tag{91}$$

The limit in (90) is indeterminate since the integral goes to zero. That is, the limiting interference power $\lim c^* < \bar{K}$ and $f_I(x) \to \delta(x - \bar{K})$. Applying l'Hôpital's rule gives

$$\lim_{(N,K,B)\to\infty} h(c^*) = \lim_{(N,K,B)\to\infty} \frac{\frac{\mathrm{d}}{\mathrm{d}N}\int_0^{c^*} x f_{I;N}(x)\, \mathrm{d}x}{\frac{\mathrm{d}2^{-\bar{B}N}}{\mathrm{d}N}} \tag{92}$$

$$= \lim_{(N,K,B)\to\infty} \frac{c^* f_{I;N}(c^*) \frac{\mathrm{d}c^*}{\mathrm{d}N}}{\frac{\mathrm{d}2^{-\bar{B}N}}{\mathrm{d}N}} \tag{93}$$

$$= \lim_{(N,K,B)\to\infty} \frac{c^* f_{I;N}(c^*) \frac{1}{f_{I;N}(c^*)} \frac{\mathrm{d}2^{-\bar{B}N}}{\mathrm{d}N}}{\frac{\mathrm{d}2^{-\bar{B}N}}{\mathrm{d}N}} \tag{94}$$

$$= \lim_{(N,K,B)\to\infty} c^* \tag{95}$$

where c^* is given by (89). This establishes (81), and shows that the entries of the optimal codebook are independent.

It remains to show that each entry of the codebook should be chosen from an isotropic distribution. This follows from the observation that the optimal signature vector $\boldsymbol{s}_{\mathsf{opt}}$, i.e., the eigenvector of $\boldsymbol{R}_1$ corresponding to the minimum eigenvalue, is asymptotically isotropically distributed [17]. Namely, let $\mathcal{S}$ denote the space of all N-dimensional, unit-norm vectors, and let $\delta\mathcal{S}_1$ and $\delta\mathcal{S}_2$ denote small nonoverlapping cones in $\mathcal{S}$ with the same volume. Because $\boldsymbol{s}_{\mathsf{opt}}$ is isotropically distributed

$$\Pr\{\boldsymbol{s}_{\mathsf{opt}} \in \delta\mathcal{S}_1\} = \Pr\{\boldsymbol{s}_{\mathsf{opt}} \in \delta\mathcal{S}_2\}.$$

Suppose that, according to the distribution $F_{I;N}(\cdot)$, we allocate $\delta B_j = (\delta\bar{B}_j)N$ vectors to $\delta\mathcal{S}_j, j = 1, 2$. Conditioning on $\boldsymbol{s}_{\mathsf{opt}} \in \delta\mathcal{S}_1 \cup \delta\mathcal{S}_2$, so that $\Pr\{\boldsymbol{s}_{\mathsf{opt}} \in \delta\mathcal{S}_j\} = 1/2$, the interference power is $[W_1^\infty(\delta\bar{B}_1) + W_2^\infty(\delta\bar{B}_2)]/2$, where W_j^∞ denotes the asymptotic interference conditioned on $\Pr\{\boldsymbol{s}_{\mathsf{opt}} \in \delta\mathcal{S}_j\}$. We wish to minimize this over $(\delta\bar{B}_1, \delta\bar{B}_2)$ subject to $\delta\bar{B}_1 + \delta\bar{B}_2$ being constant.

Because $\boldsymbol{s}_{\mathsf{opt}}$ is isotropically distributed, $W_1^\infty = W_2^\infty$. Furthermore, both $W_j^\infty(\delta\bar{B}_j)$ and $|\partial W_j^\infty/\partial(\delta\bar{B}_j)|$ are decreasing with $\delta\bar{B}_j$ (i.e., the marginal reduction in interference diminishes as new vectors are added to the codebook $\mathcal{V}$), so that the minimum occurs at $\delta\bar{B}_1 = \delta\bar{B}_2$. Consequently, the distribution of vectors $\boldsymbol{v}_k$ in the codebook should be uniformly distributed over the unit sphere, in which case $F_{I;N}(\cdot)$ becomes the distribution $F_{X|\boldsymbol{S}_1}(\cdot)$ (i.e., (56) when $K \geq N$).

C. Proof of Theorem 3

Here we prove the theorem for $K < N$. The proof for $K \geq N$ is similar to the proof of Theorem 1 and is therefore omitted. The following proof is more involved than that for the matched-filter receiver since an explicit expression for the cdf of $\beta_{\mathsf{mmse;j}}$ given $\boldsymbol{S}_1$ is unavailable. Let $Z = \max_j\{\beta_{\mathsf{mmse;j}}\}$ given $\boldsymbol{S}_1$, and let $F_{Z|\boldsymbol{S}_1}$ denote its cdf. As $n = 2^B \to \infty$, the following lemma states that $(Z - a_n)/b_n$ converges weakly to the nondegenerate Gumbel random variable H with cdf F_H.

Lemma 3: There exist a_n and b_n such that as $n \to \infty$

$$F_{Z\,|\,\boldsymbol{S}_1}(a_n + b_n x) \longrightarrow F_H(x) \tag{96}$$

for all x, where

$$F_H(x) = \exp\left(-\mathrm{e}^{-x(1+\delta_N)}\right) \tag{97}$$

and $\delta_N \to 0$ as $N \to \infty$.

Proof: This follows from [32, Theorem 2.1.3]. We again observe that the random variables $\{\beta_{\mathsf{mmse;j}}\}$ for a given $\boldsymbol{S}_1$ are i.i.d. Following the proof of [32, Theorem 2.1.3], it is sufficient to show that

$$\lim_{t\to\omega_G} \frac{1 - G_{\beta\,|\,\boldsymbol{S}_1}(t + xR(t))}{1 - G_{\beta\,|\,\boldsymbol{S}_1}(t)} = \exp(-x(1 + \delta_N)) \tag{98}$$

where

$$R(t) = \frac{1}{1 - G_{\beta\,|\,\boldsymbol{S}_1}(t)} \int_t^{\omega_G} (1 - G_{\beta\,|\,\boldsymbol{S}_1}(y))\, \mathrm{d}y \tag{99}$$

and $\omega_G = \sup\{x : G_{\beta|\boldsymbol{S}_1}(x) < 1\}$.

Let

$$u(x) = \frac{1 - G_{\beta\,|\,\boldsymbol{S}_1}(x)}{g_{\beta\,|\,\boldsymbol{S}_1}(x)}. \tag{100}$$

From [32, p. 103]

$$\frac{1-G_{\beta|\boldsymbol{S}_1}(t+xR(t))}{1-G_{\beta|\boldsymbol{S}_1}(t)} = \exp\left(-x\frac{R(t)}{u(\xi)}\right), \quad \text{for } t<\xi<t+xR(t) \tag{101}$$

so that it remains to show that

$$\lim_{t\to\omega_G}\frac{R(t)}{u(\xi)} = 1+\delta_N. \tag{102}$$

By writing

$$\frac{R(t)}{u(\xi)} = \frac{R(t)}{u(t)}\frac{u(t)}{u(\xi)} \tag{103}$$

we can evaluate the limit of each fraction on the right-hand side. Following the proof of Theorem 2.1.3 in [32], it can be shown that for any x

$$\lim_{t\to\omega_G}\frac{u(t)}{u(\xi)} = 1 - x\lim_{t\to\omega_G}\left[\frac{R(t)}{u(t)}u'(\eta)\right] \tag{104}$$

where $t<\eta<\xi$ and that

$$\lim_{t\to\omega_G}\frac{R(t)}{u(t)} = \lim_{t\to\omega_G}\frac{1}{1-u'(t)}. \tag{105}$$

We show in Appendix D that

$$u'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1-G_{\beta\,|\,\boldsymbol{S}_1}(x)}{g_{\beta\,|\,\boldsymbol{S}_1}(x)}\right] = -\frac{1}{N-1}+\epsilon(x) \tag{106}$$

where $\epsilon(x)\to 0$ as $x\to\omega_G$. Combining (103)–(105) implies (102), which in turn implies (98), (99), and the lemma. □

From [32, Theorem 2.1.3], the normalizing sequences are

$$a_n = G^{-1}_{\beta\,|\,\boldsymbol{S}_1}\left(1-\frac{1}{2^B}\right) \tag{107}$$

$$b_n = G^{-1}_{\beta\,|\,\boldsymbol{S}_1}\left(1-\frac{1}{\mathrm{e}2^B}\right) - G^{-1}_{\beta\,|\,\boldsymbol{S}_1}\left(1-\frac{1}{2^B}\right). \tag{108}$$

Therefore,

$$\lim_{(N,K,B)\to\infty} E[Z] = \lim_{(N,K,B)\to\infty} a_n + b_n E[H] \tag{109}$$

$$= \lim_{(N,K,B)\to\infty} G^{-1}_{\beta\,|\,\boldsymbol{S}_1}\left(1-\frac{1}{2^B}\right) \tag{110}$$

where we have used the fact that $E[H]$ is bounded and $b_n\to 0$ as $(N,K,B)\to\infty$. Furthermore, as in the proof of Theorem 1, it is straightforward to show that $E[(Z-E[Z])^2]\to 0$ as $N\to\infty$.

To prove the inequality in the theorem, we define $\beta_{j;N}$ as the SINR corresponding to $\boldsymbol{v}_j\in\mathcal{V}_N$ for a given $\boldsymbol{S}_1$, and $F_{\beta;N}$ as the associated cdf, which is independent of j. (Note that $F_{\beta;N}$ corresponds to $\boldsymbol{v}_j$ being chosen from an arbitrary cdf, whereas $G_{\beta\,|\,\boldsymbol{S}_1}(\cdot)$, defined earlier, assumes that $\boldsymbol{v}_j$ is isotropically distributed.) As in the proof of Theorem 2, we first show that

$$\lim_{(N,K,B)\to\infty} E[\max_{1\le j\le 2^B}\beta_{j;N}] \le \lim_{(N,K,B)\to\infty} F^{-1}_{\beta;N}\left(1-\frac{1}{2^B}\right) \tag{111}$$

which implies that the codebook entries should be independent. We then show that each vector should be isotropically distributed.

Since $\beta_{j;N}\ge 0$, we have for $\forall c\in\mathbb{R}$

$$\max_{1\le j\le 2^B}\beta_{j;N} \le c + \sum_{j=1}^{2^B}(\beta_{j;N}-c)^+. \tag{112}$$

Taking expectation and the large system limit gives

$$\lim_{(N,K,B)\to\infty} E[\max_{1\le j\le 2^B}\beta_{j;N}] \le c + \lim_{(N,K,B)\to\infty} 2^B E[(\beta_{j;N}-c)^+] \tag{113}$$

and the c which minimizes the right-hand side is

$$c^* = F^{-1}_{\beta;N}\left(1-\frac{1}{2^B}\right). \tag{114}$$

Substituting for c^* in (113) gives (111).

As with the matched-filter receiver, the optimal signature vector with the MMSE receiver, i.e., the eigenvector of $\boldsymbol{R}_1^{-1}$ corresponding to the maximum eigenvalue, is again isotropically distributed as $(K,N)\to\infty$. Following an analogous argument as for the matched filter shows that the asymptotic SINR is maximized when each $\boldsymbol{v}_j$ is uniformly distributed over the unit sphere. (Here we use the fact that the asymptotic SINR $\beta^\infty(\delta\bar{B})$ is concave increasing, where each $\boldsymbol{v}_j$ is constrained to lie in a small cone $\delta\mathcal{S}$.) Hence,

$$\lim_{(K,N,B)\to\infty} F^{-1}_{\beta;N}\left(1-\frac{1}{2^B}\right) \le \lim_{(K,N,B)\to\infty} G^{-1}_{\beta\,|\,\boldsymbol{S}_1}\left(1-\frac{1}{2^B}\right) \tag{115}$$

where $G_{\beta\,|\,\boldsymbol{S}_1}(\cdot)$ is the cdf of $\beta_{j;N}$ with RVQ.

D. Derivation of (106)

For $\bar{K}\le 1$

$$1-G_{\beta\,|\,\boldsymbol{S}_1}(x) = \Pr\left\{x\le\beta_{\mathsf{mmse;j}}\le\frac{1}{\sigma_n^2}\right\}. \tag{116}$$

We are interested in the behavior of this function as $x\to 1/\sigma_n^2$. We express

$$\beta_{\mathsf{mmse;j}} = \sum_{i=1}^N \lambda_i(\boldsymbol{v}_j^\dagger\boldsymbol{u}_i)^2 \tag{117}$$

$$= \sum_{i=1}^N \lambda_i\|\boldsymbol{v}_j\|^2\|\boldsymbol{u}_i\|^2\cos^2\theta_i \tag{118}$$

$$= \sum_{i=1}^N \lambda_i\cos^2\theta_i \tag{119}$$

where $\{\lambda_i\}$ and $\{\boldsymbol{u}_i\}$ are the eigenvalues and eigenvectors of $\boldsymbol{R}_1^{-1}$, respectively, and θ_i is the angle between $\boldsymbol{v}_j$ and $\boldsymbol{u}_i$. Each eigenvector is isotropically distributed [17], i.e., uniformly distributed over the unit sphere as $N\to\infty$. For a given $\boldsymbol{u}_i$ and N, the probability that $0<\theta_i<t$ is the area of the surface of the corresponding N-dimensional spherical cap normalized by the total surface area of the sphere, i.e.,

$$\Pr\{0\le\theta_i\le t\} = \frac{1}{2}(1-\cos^2 t)^{N-1}+\epsilon_N \tag{120}$$

where $\epsilon_N \to 0$ as $N \to \infty$. (A similar computation in a related context is given in [20].) Conditioned on $\boldsymbol{u}_i$ and $\lambda_i = \lambda$, we have

$$\Pr\{\lambda \cos^2\theta_i \le y\} = 1 - \left(1 - \frac{y}{\lambda}\right)^{N-1} + \epsilon_N. \tag{121}$$

Now $\beta_{\mathsf{mmse;j}}$ is close to $1/\sigma_n^2$ when $\boldsymbol{v}_j$ is close to an eigenvector with corresponding eigenvalue $1/\sigma_n^2$. Since there are $N-K$ eigenvectors with this eigenvalue, and because being in the vicinity of any one of those eigenvectors are mutually exclusive events as $N \to \infty$, we have

$$\Pr\left\{ x \le \beta_{\mathsf{mmse;j}} \le \frac{1}{\sigma_n^2} \middle| \boldsymbol{S}_1 \right\} = (1-\bar{K})\frac{1}{N}(1-x\sigma_n^2)^{N-1} + \epsilon_N + \tilde{\epsilon}_N(x) \tag{122}$$

where $\tilde{\epsilon}_N(x) \to 0$ uniformly as $x \to 1/\sigma_n^2$. Differentiating gives the pdf

$$g_{\beta\,|\,\boldsymbol{S}_1}(x) = (1-\bar{K})\sigma_n^2 \frac{N-1}{N}\left(1-x\sigma_n^2\right)^{N-2} + \tilde{\epsilon}'_N(x) \tag{123}$$

where $\tilde{\epsilon}'_N(x) \to 0$ as $x \to 1/\sigma_n^2$. Therefore, we can write

$$\frac{1-G_{\beta|\boldsymbol{S}_1}(x)}{g_{\beta|\boldsymbol{S}_1}(x)} = \frac{1-x\sigma_n^2}{\sigma_n^2(N-1)} + \check{\epsilon}_N + \check{\epsilon}(x) \tag{124}$$

where $\check{\epsilon}_N \to 0$ as $N \to \infty$, and $\check{\epsilon}(x) \to 0$ as $x \to \omega_G$. Differentiating gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1-G_{\beta\,|\,\boldsymbol{S}_1}(x)}{g_{\beta\,|\,\boldsymbol{S}_1}(x)}\right] = -\frac{1}{N-1} + \check{\epsilon}'(x) \tag{125}$$

where $\check{\epsilon}'(x) \to 0$ as $x \to \omega_G$.

E. Proof of Theorem 4

In what follows, we determine the limiting eigenvalue distribution of the matrix $\boldsymbol{A} = \boldsymbol{R}_1^{-1}\boldsymbol{F}_1\boldsymbol{F}_1^\dagger$, which has the same maximum eigenvalue as $\boldsymbol{F}_1^\dagger\boldsymbol{R}_1^{-1}\boldsymbol{F}_1$. This derivation relies on results from free probability theory and its application to large random matrices. (See [34], [35], [41]–[43] for a more detailed treatment of this topic.)

The empirical eigenvalue distribution (e.e.d.) of an $N \times N$ matrix $\boldsymbol{M}$ with eigenvalues $\lambda_1, \ldots, \lambda_N$ is defined as $|\mathcal{A}(x)|/N$ where $\mathcal{A}(x) = \{\lambda_j : \lambda_j < x\}$. As $N \to \infty$, the e.e.d. for certain classes of random matrices converges weakly in distribution to a deterministic distribution, which we denote as $G_{\boldsymbol{M}}(\,\cdot\,)$. In particular, this is true for $\boldsymbol{A}$ as $(D,K,N) \to \infty$ with fixed D/K and K/N.

We first note that from [34, Proposition 4.3.9], the family $\{\boldsymbol{R}_1^{-1}, \boldsymbol{F}_1\boldsymbol{F}_1^\dagger\}$ is asymptotically free almost everywhere as $N \to \infty$. Hence, we have

$$G_{\boldsymbol{A}} = G_{\boldsymbol{R}_1^{-1}} \boxtimes G_{\boldsymbol{F}_1\boldsymbol{F}_1^\dagger}$$

where $\boxtimes$ denotes free multiplicative convolution. To evaluate $G_{\boldsymbol{A}}$, we first note that the eigenvalues of $\boldsymbol{F}_1\boldsymbol{F}_1^\dagger$ are either zero or one so that

$$\frac{\mathrm{d}G_{\boldsymbol{F}_1\boldsymbol{F}_1^\dagger}(x)}{\mathrm{d}x} = \bar{D}\bar{K}\delta(x-1) + (1-\bar{D}\bar{K})\delta(x). \tag{126}$$

For a distribution $G(\,\cdot\,)$, let

$$\psi_G(z) = \int \frac{zt}{1-zt}\mathrm{d}G(t) \tag{127}$$

which is the z-transform of the moments of $G(t)$. We will use the shorthand notation $\psi_{\boldsymbol{M}}$ to denote $\psi_{G_{\boldsymbol{M}}}$. Evaluating the free multiplicative convolution gives

$$\psi_{\boldsymbol{A}}(z) = \psi_{\boldsymbol{R}_1^{-1}}\left(z\frac{\bar{D}\bar{K} + \psi_{\boldsymbol{A}}(z)}{1+\psi_{\boldsymbol{A}}(z)}\right). \tag{128}$$

We skip the details, since a similar result appears in [42, eq. (29)].

To solve for $\psi_{\boldsymbol{A}}(z)$ in (128), we must obtain an expression for $\psi_{\boldsymbol{R}_1^{-1}}(z)$. Using the fact that the eigenvalues of $\boldsymbol{R}_1$ are the inverse eigenvalues of $\boldsymbol{R}_1^{-1}$, it is straightforward to show that

$$\psi_{\boldsymbol{R}_1^{-1}}(z) = z\int \frac{1}{t-z}\mathrm{d}G_{\boldsymbol{R}}(t) = zm_{\boldsymbol{R}}(z) \tag{129}$$

where $m_{\boldsymbol{R}}(z) = \int \frac{1}{t-z}\mathrm{d}G_{\boldsymbol{R}}(t)$ is the Stieltjes transform of $G_{\boldsymbol{R}}(\,\cdot\,)$, and satisfies the fixed-point equation [29, Theorem 4.1]

$$m_{\boldsymbol{R}}(z) = \frac{1}{\sigma_n^2 - z + \frac{\bar{K}}{1+m_{\boldsymbol{R}}(z)}}. \tag{130}$$

Substituting (130) into (129), we have

$$\psi_{\boldsymbol{R}_1^{-1}}(z) = \frac{1}{\frac{\sigma_n^2}{z} - 1 + \frac{\bar{K}}{z+\psi_{\boldsymbol{R}_1^{-1}}(z)}}. \tag{131}$$

We also note that

$$\psi_{\boldsymbol{A}}(z^{-1}) = \int \frac{z^{-1}t}{1-z^{-1}t}\mathrm{d}G_{\boldsymbol{A}}(t) = -zm_{\boldsymbol{A}}(z) - 1. \tag{132}$$

Combining (128), (131), and (132) gives, after some algebraic manipulation

$$a_3m_{\boldsymbol{A}}^3 + a_2m_{\boldsymbol{A}}^2 + a_1m_{\boldsymbol{A}} + a_0 = 0 \tag{133}$$

where

$$a_3 = \sigma_n^2z^5 - z^4 \tag{134}$$
$$a_2 = 2\sigma_n^2z^4 + (\bar{D}\bar{K} - 2 - \sigma_n^2 - \bar{K})z^3 + z^2 \tag{135}$$
$$a_1 = \sigma_n^2z^3 + (\bar{K}^2\bar{D} - 1 + \bar{D}\bar{K} - 2\sigma_n^2 - 2\bar{K} + \sigma_n^2\bar{D}\bar{K})z^2 + (2-2\bar{D}\bar{K})z \tag{136}$$
$$a_0 = (\bar{K}^2\bar{D} - \sigma_n^2 - \bar{K} + \sigma_n^2\bar{D}\bar{K})z + (\bar{D}^2\bar{K}^2 + 1 - 2\bar{D}\bar{K}). \tag{137}$$

If $\mathrm{Im}(m_{\boldsymbol{A}}(z))$ is continuous at $z = t_0 + j \cdot 0$, then the corresponding distribution $G_{\boldsymbol{A}}(\cdot)$ is unique, and the corresponding pdf at $t = t_0$ is given by [31]

$$\left.\frac{\mathrm{d}G_{\boldsymbol{A}}(t)}{\mathrm{d}t}\right|_{t=t_0} = \frac{1}{\pi} \lim_{\zeta \to 0} \mathrm{Im}(m_{\boldsymbol{A}}(t_0 + j\zeta)). \tag{138}$$

If the maximum eigenvalue in the large system limit is t_0, then $\mathrm{d}G_{\boldsymbol{A}}(t)/\mathrm{d}t$ evaluated at t_0 must be zero. The right-hand side of (138) is zero when $m_{\boldsymbol{A}}(t_0 + j0)$ is real. That implies that the root of (133), which gives $m_{\boldsymbol{A}}$, must be real. The cubic equation (133) has only real roots when the following equation holds [44]:

$$\left(\frac{3a_2 - a_1^2}{9}\right)^3 + \left(\frac{9a_1a_2 - 27a_3 - 2a_1^3}{54}\right)^2 = 0. \tag{139}$$

Simplifying the preceding equation leads to the quartic equation (33) in Theorem 4, where the constants are as follows:

$$b_4 = -\sigma_n^4 \bar{K}^2 + 2\sigma_n^4 \left(1 - \sigma_n^2\right) \bar{K} - \sigma_n^4 \left(1 + 2\sigma_n^2 + \sigma_n^4\right) \tag{140}$$

$$\begin{aligned} b_3 = {} & 4\sigma_n^2 \bar{D} \bar{K}^4 + 2\sigma_n^2 \left(-1 - 5\bar{D} + 6\sigma_n^2 \bar{D}\right) \bar{K}^3 \\ & + 2\sigma_n^2 \left(-\sigma_n^2 \bar{D} + 3 + 4\bar{D} - 3\sigma_n^2 + 6\sigma_n^4 \bar{D}\right) \bar{K}^2 \\ & + 2\sigma_n^2 \left(-3 + 4\sigma_n^4 \bar{D} + \sigma_n^2 \bar{D} - \bar{D} + 2\sigma_n^2 - 3\sigma_n^4 \right. \\ & \left. + 2\sigma_n^6 \bar{D}\right) \bar{K} + 2\sigma_n^2 \left(1 + \sigma_n^2 - \sigma_n^4 - \sigma_n^6\right) \end{aligned} \tag{141}$$

$$\begin{aligned} b_2 = {} & -(\bar{D} + 1)^2 \bar{K}^4 \\ & + 2\left(2 + 3\bar{D} + \bar{D}^2 - 2\sigma_n^2 - 6\sigma_n^2 \bar{D} - 10\sigma_n^2 \bar{D}^2\right) \bar{K}^3 \\ & + \left(-6 - 6\bar{D} - \bar{D}^2 + 10\sigma_n^2 + 22\sigma_n^2 \bar{D} + 8\sigma_n^2 \bar{D}^2 \right. \\ & \left. - 6\sigma_n^4 - 18\sigma_n^4 \bar{D} + 8\sigma_n^4 \bar{D}^2\right) \bar{K}^2 + 2\left(2 + \bar{D} - 4\sigma_n^2 \right. \\ & \left. - 5\sigma_n^2 \bar{D} + 4\sigma_n^4 - 10\sigma_n^4 \bar{D} - 2\sigma_n^6 - 4\sigma_n^6 \bar{D}\right) \bar{K} \\ & - 1 + 2\sigma_n^2 + 6\sigma_n^4 + 2\sigma_n^6 - \sigma_n^8 \end{aligned} \tag{142}$$

$$\begin{aligned} b_1 = {} & 2(1 + 4\bar{D} + 5\bar{D}^2 + 2\bar{D}^3)\bar{K}^3 \\ & + 2\left(-3 - 8\bar{D} - 5\bar{D}^2 + 3\sigma_n^2 + 5\sigma_n^2 \bar{D} - 4\sigma_n^2 \bar{D}^2\right) \bar{K}^2 \\ & + 2\left(3 + 4\bar{D} - 2\sigma_n^2 + 5\sigma_n^2 \bar{D} + 3\sigma_n^4 + \sigma_n^4 \bar{D}\right) \bar{K} \\ & + 2\left(-1 - \sigma_n^2 + \sigma_n^4 + \sigma_n^6\right) \end{aligned} \tag{143}$$

$$\begin{aligned} b_0 = {} & -(\bar{D} + 1)^2 \bar{K}^2 + 2\left(1 + \bar{D} - \sigma_n^2 + \sigma_n^2 \bar{D}\right) \bar{K} \\ & - 1 - 2\sigma_n^2 - \sigma_n^4. \end{aligned} \tag{144}$$

F. Derivation of (38)

Substituting (34) into (37), and expanding the interference term in the denominator gives

$$\begin{aligned} \hat{\boldsymbol{\alpha}}_1^\dagger \boldsymbol{F}_1^\dagger \boldsymbol{S}_1 \boldsymbol{S}_1^\dagger \boldsymbol{F}_1 \hat{\boldsymbol{\alpha}}_1 = {} & \boldsymbol{\alpha}_1^\dagger \boldsymbol{F}_1^\dagger \boldsymbol{S}_1 \boldsymbol{S}_1^\dagger \boldsymbol{F}_1 \boldsymbol{\alpha}_1 \\ & + 2\Delta\boldsymbol{\alpha}_1^\dagger \boldsymbol{F}_1^\dagger \boldsymbol{S}_1 \boldsymbol{S}_1^\dagger \boldsymbol{F}_1 \boldsymbol{\alpha}_1 \\ & + \Delta\boldsymbol{\alpha}_1^\dagger \boldsymbol{F}_1^\dagger \boldsymbol{S}_1 \boldsymbol{S}_1^\dagger \boldsymbol{F}_1 \Delta\boldsymbol{\alpha}_1. \end{aligned} \tag{145}$$

Each term can be evaluated as follows. First, since $\boldsymbol{\alpha}_1$ is the eigenvector of $\boldsymbol{F}_1^\dagger \boldsymbol{S}_1 \boldsymbol{S}_1^\dagger \boldsymbol{F}_1$ corresponding to the minimum eigenvalue, the first term in (145) converges to [15]

$$\lim_{(K,N,D)\to\infty} \boldsymbol{\alpha}_1^\dagger \boldsymbol{F}_1^\dagger \boldsymbol{S}_1 \boldsymbol{S}_1^\dagger \boldsymbol{F}_1 \boldsymbol{\alpha}_1 = (1 - \sqrt{\bar{D}})^2 \bar{K} \tag{146}$$

for $\bar{K} \leq 1$.

Next, we take the large system limit of the cross term in (145), which gives

$$\begin{aligned} & \lim_{(K,N,D,B)\to\infty} \Delta\boldsymbol{\alpha}_1^\dagger \boldsymbol{F}_1^\dagger \boldsymbol{S}_1 \boldsymbol{S}_1^\dagger \boldsymbol{F}_1 \boldsymbol{\alpha}_1 \\ & \qquad = \left(1 - \sqrt{\bar{D}}\right)^2 \bar{K} \lim_{(K,N,D,B)\to\infty} \Delta\boldsymbol{\alpha}_1^\dagger \boldsymbol{\alpha}_1. \end{aligned} \tag{147}$$

To evaluate (147), we examine the correlation between the coefficients $[\boldsymbol{\alpha}_1]_i$ and $[\Delta\boldsymbol{\alpha}_1]_i$. To simplify the notation, we will refer to these as α_i and $\Delta\alpha_i$, respectively. When the quantizer is optimal in the MSE sense, the quantized output power is given by [45]

$$E[(\alpha_i + \Delta\alpha_i)^2] = E\left[\alpha_i^2\right] - E\left[\Delta\alpha_i^2\right]. \tag{148}$$

Expanding the square on the left-hand side, we have

$$E[\alpha_i \Delta\alpha_i] = -E\left[\Delta\alpha_i^2\right] \tag{149}$$

so that

$$\begin{aligned} & \lim_{(K,N,D,B)\to\infty} E[\Delta\boldsymbol{\alpha}_1^\dagger \boldsymbol{F}_1^\dagger \boldsymbol{S}_1 \boldsymbol{S}_1^\dagger \boldsymbol{F}_1 \boldsymbol{\alpha}_1] \\ & \qquad = -\left(1 - \sqrt{\bar{D}}\right)^2 \bar{K} E[\|\Delta\bar{\boldsymbol{\alpha}}_1\|^2] \end{aligned} \tag{150}$$

where $\Delta\bar{\boldsymbol{\alpha}}_1 = \lim_{(N,D,B)\to\infty} \Delta\boldsymbol{\alpha}_1$.

To evaluate the last term in (145), we make the approximation that $\Delta\boldsymbol{\alpha}_1$ is independent of matrix $\boldsymbol{F}_1^\dagger \boldsymbol{S}_1 \boldsymbol{S}_1^\dagger \boldsymbol{F}_1$. This approximation is accurate for moderate to large normalized feedback B/D. For small B/D, $\Delta\boldsymbol{\alpha}_1$ strongly depends on $\boldsymbol{\alpha}_1$, which, in turn, depends on $\boldsymbol{F}_1^\dagger \boldsymbol{S}_1 \boldsymbol{S}_1^\dagger \boldsymbol{F}_1$. Taking the large system limit with the independence assumption gives

$$\lim_{(K,N,D,B)\to\infty} E[\Delta\boldsymbol{\alpha}_1^\dagger \boldsymbol{F}_1^\dagger \boldsymbol{S}_1 \boldsymbol{S}_1^\dagger \boldsymbol{F}_1 \Delta\boldsymbol{\alpha}_1] \approx \bar{K} E[\|\Delta\bar{\boldsymbol{\alpha}}_1\|^2]. \tag{151}$$

The quantization noise power can be computed as

$$E[\|\Delta\boldsymbol{\alpha}_1\|^2] = D \sum_{l=1}^{m-1} \int_{x_l}^{x_{l+1}} (\hat{y}_l - y)^2 f_\alpha(y)\,\mathrm{d}y \tag{152}$$

where the number of quantization levels $m = 2^{B/D}$, $\{x_l\}$, $1 \leq l \leq m$, are the decision thresholds, $\hat{y}_l$ is the quantized value, and f_α is the pdf of α_i. As stated earlier, we will approximate f_α as Gaussian with zero mean and variance $1/D$. For large m, the quantization noise power in (152) is given by [37]

$$E[\|\Delta\boldsymbol{\alpha}_1\|^2] = \frac{\sqrt{3}\pi}{2^{2B/D+1}} + O\left(\frac{1}{m^2}\right) \tag{153}$$

$$\approx q(\bar{D}, \bar{K}, \bar{B}). \tag{154}$$

Combining (37), (146), (150), (151), and (154) gives the approximation (38).

G. Derivation of (51)

Using any receiver other than the MMSE receiver can only increase the sum MSE. Consider the following receiver:

$$\boldsymbol{c}_k = \boldsymbol{S}(\boldsymbol{S}^\dagger \boldsymbol{R} \boldsymbol{S})^{-1} \boldsymbol{S}^\dagger \boldsymbol{s}_k \tag{155}$$

which is the optimal receiver constrained to lie in the subspace spanned by the columns of $\boldsymbol{S}$. The corresponding MSE is

$$\bar{\mathcal{M}}_k = \boldsymbol{c}_k^\dagger \boldsymbol{R} \boldsymbol{c}_k - 2\boldsymbol{c}_k^\dagger \boldsymbol{s}_k \tag{156}$$

$$= 1 - \boldsymbol{s}_k^\dagger \boldsymbol{S}(\boldsymbol{S}^\dagger \boldsymbol{R} \boldsymbol{S})^{-1}\boldsymbol{S}^\dagger \boldsymbol{s}_k \tag{157}$$

and the sum MSE can be written as

$$\bar{\mathcal{M}} = \frac{1}{K_a}\sum_{i=1}^{K_a} \bar{\mathcal{M}}_i \tag{158}$$

$$= \frac{1}{K_a}\mathrm{tr}\{\boldsymbol{I} - \boldsymbol{S}^\dagger \boldsymbol{S}(\boldsymbol{S}^\dagger \boldsymbol{R}\boldsymbol{S})^{-1}\boldsymbol{S}^\dagger \boldsymbol{S}\} \tag{159}$$

$$= \frac{1}{K_a}\mathrm{tr}\{\boldsymbol{I} - (\boldsymbol{S}^\dagger \boldsymbol{R}\boldsymbol{S})^{-1}\} \tag{160}$$

where the last equality holds because $\boldsymbol{S}$ is orthonormal. Let $\boldsymbol{R}_S = \boldsymbol{P}\boldsymbol{P}^\dagger + \sigma_n^2 \boldsymbol{I}$ (i.e., the covariance matrix excluding the signatures in $\boldsymbol{S}$), and η_i denote the ith eigenvalue of the matrix $\boldsymbol{S}^\dagger \boldsymbol{R}_S \boldsymbol{S}$. We have

$$\mathcal{M} \le \bar{\mathcal{M}} \tag{161}$$

$$= \frac{1}{K_a}\mathrm{tr}\{\boldsymbol{I} - (\boldsymbol{S}^\dagger \boldsymbol{R}\boldsymbol{S})^{-1}\} \tag{162}$$

$$= \frac{1}{K_a}\mathrm{tr}\{\boldsymbol{I} - (\boldsymbol{I} + \boldsymbol{S}^\dagger \boldsymbol{R}_S\boldsymbol{S})^{-1}\} \tag{163}$$

$$= 1 - \frac{1}{K_a}\sum_i \frac{1}{1+\eta_i} \tag{164}$$

$$\le 1 - \frac{1}{1 + \frac{1}{K_a}\sum_i \eta_i} \tag{165}$$

$$= 1 - \frac{1}{1 + \frac{1}{K_a}\mathrm{tr}\{\boldsymbol{S}^\dagger \boldsymbol{R}_S \boldsymbol{S}\}} \tag{166}$$

$$= 1 - \frac{1}{1 + \sigma_n^2 + \frac{1}{K_a}\mathrm{tr}\{\boldsymbol{P}^\dagger \boldsymbol{S}\boldsymbol{S}^\dagger \boldsymbol{P}\}}. \tag{167}$$

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