ACE 261
Fall 2002
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Lecture 4

Introduction to Probability

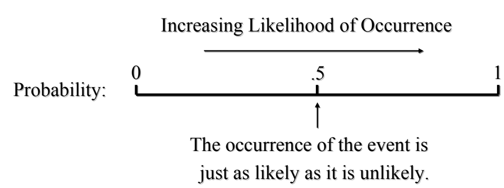
Outline

- Experiments, counting rules, and assigning probabilities
- Events and their probability
- Some basic relationships of probability
- Conditional probability
- Bayes' theorem

Probability

- Probability is a numerical measure of the likelihood that an event will occur.
- Probability values are always assigned on a scale from 0 to 1.
  - A probability near 0 indicates an event is very unlikely to occur.
  - A probability near 1 indicates an event is almost certain to occur.
  - A probability of 0.5 indicates the occurrence of the event is just as likely as it is unlikely.

Probability as a Numerical Measure of the Likelihood of Occurrence



An experiment and its sample space

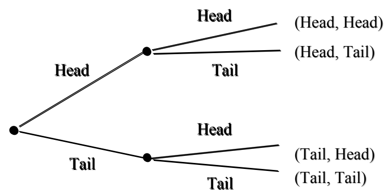
- An experiment is any process that generates well-defined outcomes.
  - Example: toss a coin
  - play a football game
- The sample space for an experiment is the set of all experimental outcomes.
  S={Head, tail}
  S={Win, lose or tie}
- A sample point is an element of the sample space, any one particular experimental outcome.
  - Head
  - Tie

Multiple-Step Experiments

- A multiple-step experiment consists of a sequence of k steps in which there are n_1 possible results for the first step, n_2 possible results for the second step, and so on.
- The total number of experimental outcomes is given by $(n_1)(n_2) \ldots (n_k)$.
- Example: with two steps and two results per step, there are four outcomes.
- A helpful graphical representation of a multiple-step experiment is a tree diagram.

Tree diagram

Toss a coin (Step 1) | Toss a coin (Step 2) | Experimental Outcomes



Combinations

- A combination is a group of *n* objects that are selected from a set of *N* objects.
- The number of combinations of *N* objects taken *n* at a time is:

$$C_n^N = \binom{N}{n} = \frac{N!}{n!(N-n)!}$$

where $N! = N(N-1)(N-2)\ldots(2)(1)$

$n! = n(n-1)(n-2)\ldots(2)(1)$

$0! = 1$

Combinations: a lottery example

- Let's play this lottery: select two numbers between 1 and 5.
- What is the number of all possible combinations of 2 numbers out of 5?

$$C_2^5 = \binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{(5)(4)(3)(2)(1)}{(2)(1)(3)(2)(1)} = 10$$

- These combinations are: (1,2) (1,3) (1,4) (1,5) (2,3) (2,4) (2,5) (3,4) (3,5) (4,5).
- The probability of winning this lottery is 1/10 = 0.1.

Permutations

- A permutation is a group of *n* objects that are selected from a set of *N* objects where the order of selection is important.
- So, (1,3) and (3,1) are the same combination but different permutations.
- Number of permutations of *N* objects taken *n* at a time is:

$$P_n^N = n!\binom{N}{n} = \frac{N!}{(N-n)!}$$

Permutations: a lottery example

- What is the number of all possible permutations of 2 numbers out of 5?

$$P_2^5 = 2!\binom{5}{2} = \frac{5!}{(5-2)!} = \frac{(5)(4)(3)(2)(1)}{(3)(2)(1)} = 20$$

- These permutations are:
  - (1,2) (1,3) (1,4) (1,5)
  - (2,1) (2,3) (2,4) (2,5)
  - (3,1) (3,2) (3,4) (3,5)
  - (4,1) (4,2) (4,3) (4,5)
  - (5,1) (5,2) (5,3) (5,4).
- The probability of winning this lottery is 1/20 = 0.05.

Assigning probabilities

- Classical method
  - Assigning probabilities based on the assumption of equally likely outcomes.
- Relative frequency method
  - Assigning probabilities based on the relative frequencies of experimentation or historical data.
- Subjective method
  - Assigning probabilities based on person's degree of belief.

Classical method

- If an experiment has n possible outcomes, this method would assign a probability of $1/n$ to each outcome.
- Example

  Experiment: Rolling a die

  Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$

  Probabilities: Each sample point has a 1/6 chance of occurring.

Relative frequency method

Daily sales of cars for a car dealer	Frequency	Relative frequency= probability
0	4	0.10
1	6	0.15
2	18	0.45
3	10	0.25
4	2	0.05
Total	40	1.00

Subjective Method

- We can use any data available as well as our experience and intuition, but ultimately a probability value should express our degree of belief that the experimental outcome will occur.
- Example: I think that there is a 60% probability that the economy will get out of recession.

Events and their probability

- An event is a collection of sample points.

  Example: Event A: an odd number after rolling a die

  $A=\{1,3,5\}$
- The probability of any event is equal to the sum of the probabilities of the sample points in the event.

  $P(A)=P(1)+P(3)+P(5)= 1/6 + 1/6 +1/6 = 1/2$
- If we can identify all the sample points of an experiment and assign a probability to each, we can compute the probability of an event.

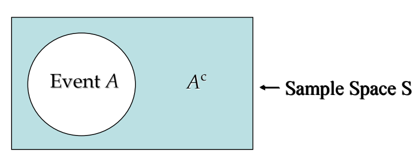
Some basic relationships of probability

- There are some basic probability relationships that can be used to compute the probability of an event without knowledge of all sample point probabilities.
  - Complement of an Event
  - Union of Two Events
  - Intersection of Two Events
  - Mutually Exclusive Events

Complement of an event

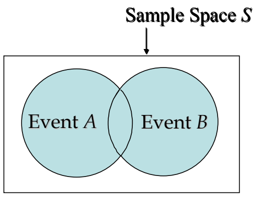
- The complement of event A is defined to be the event consisting of all sample points that are not in A.
- The complement of A is denoted by A^c.

  $P(A)+P(A^c)=1$
- The Venn diagram below illustrates the concept of a complement.



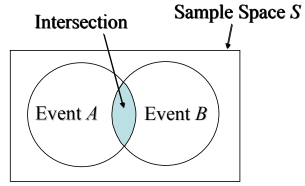
Union of two events

- The union of events *A* and *B* is the event containing all sample points that are in *A or B* or both.
- The union is denoted by $A \cup B$.



Intersection of two events

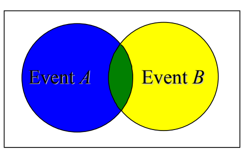
- The intersection of events *A* and *B* is the set of all sample points that are in both *A and B*.
- The intersection is denoted by $A \cap B$.
- The intersection of *A* and *B* is the area of overlap in the illustration below.



Addition law

- The addition law provides a way to compute the probability of event *A,* or *B,* or both *A* and *B* occurring.
- The addition law is:

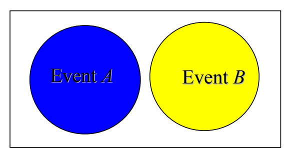
P(*A or B*) = P(*A*) + P(*B*) – P(*A and B*)



Mutually exclusive events

- Two events are mutually exclusive if the events have no sample points in common.
- This means that when one event occurs, the other cannot occur.
- This also means that P(*A and B*)=0, therefore

P(*A or B*) = P(*A*) + P(*B*) - P(*A and B*) = P(*A*) + P(*B*)



Conditional Probability

- Conditional probability is the probability of an event given that another event has occurred.
- The conditional probability of *B given A* is denoted by P(*B*|*A*).
- Conditional probability is the joint probability P(A and B) over the marginal probability P(A).

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$$

Multiplication law

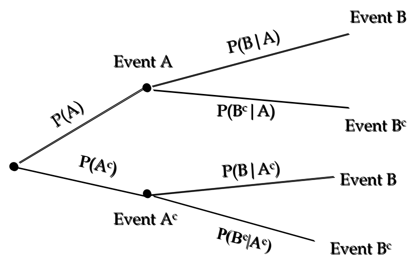
- The multiplication law provides a way to compute the probability of an intersection of two events, *A and B*.
- By re-arranging the conditional probability formula

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

- The multiplication law is written as:

$$P(A \text{ and } B) = P(B)P(A \mid B)$$

Conditional probabilities



Joint probability table

Events	B	B^c	Totals
A	P(A and B) Joint probability	P(A and B^c) Joint Probability	P(A) Marginal Probability
A^c	P(A^c and B) Joint Probability	P(A^c and B^c) Joint Probability	P(A^c) Marginal Probability
Totals	P(B) Marginal Probability	P(B^c) Marginal Probability	1

Joint probability table

	Go to grad school	Don't go to grad school	Totals
Male	0.60	0.20	0.80
Female	0.14	0.06	0.20
Totals	0.74	0.26	1

Conditional probability example

- The joint probabilities for being male/female and going/not going to grad school are given in table on the next slide.
- What is the probability of going to grad school given that you are male?
- P (go to grad school | male) = P (go to grad school and male) / P (male) = 0.6/0.8= 0.75
- P (go to grad school)=0.74
- P (go to grad school | female) = P (go to grad school and female) / P (female) =

Independent Events

- Events *A* and *B* are <u>independent</u> if P(*A*|*B*) = P(*A*) or P(B|A)=P(B).
- Therefore, the multiplication law for independent events is:

[illegible]

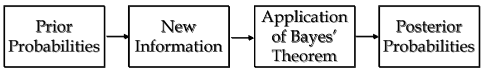
- The multiplication law also can be used as a test to see if two events are independent.

Joint probability table for independent events

	Go to grad school	Don't go to grad school	Totals
Male	=(.8)(.7)=.56	=(.8)(.3)=.24	0.8
Female	=(.2)(.7)=.14	=(.2)(.3)=.06	0.2
Totals	0.7	0.3	1

Bayes' Theorem

- Often we begin probability analysis with initial or <u>prior probabilities</u>.
- Then, from a sample, special report, or a product test we obtain some additional information.
- Given this information, we calculate revised or <u>posterior probabilities</u>.
- <u>Bayes' theorem</u> is a method to compute posterior probabilities by revising prior probabilities.



Bayes' formula

- From the conditional probability formula, we know that:

$$P(A_1 \mid B) = \frac{P(A_1 \text{ and } B)}{P(B)} = \frac{P(A_1 \text{ and } B)}{P(A_1 \text{ and } B) + P(A_2 \text{ and } B)}$$

- This is the same as:

$$P(A_1 \mid B) = \frac{P(A_1)P(B \mid A_1)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2)}$$

Bayes' theorem: a labor economics example

- Suppose that the manager of a factory hires new workers. The workers can be high effort workers with probability 80% and low effort workers with probability 20%.
- The manager wants to give higher salaries to the high effort workers but cannot recognize the effort type of each worker.
- High effort workers produce good products with 90% probability and bad products with 10% probability.
- Low effort workers produce good products with 30% probability and bad products with 70% probability.

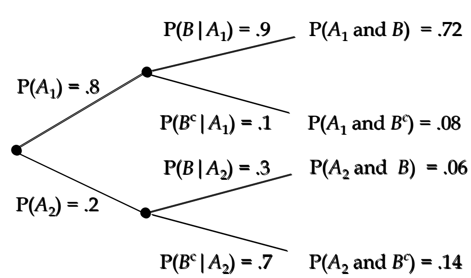
Bayes' theorem: a labor economics example

- After one period, the manager wants to give higher salaries to the high effort workers, but the manager still cannot recognize who the high effort workers are. The manager can only observe who produced good products.
- So, given that a worker produced good products, what is the probability that he is a high effort worker?

Bayes' theorem: a labor economics example

- Event A_1: high effort worker
  $P(A_1)=$
- Event A_2: low effort worker
  $P(A_2)=$
- Event B: produce good products
  - Given that the worker exerts high effort, $P(B|A_1)=$
  - Given that the worker exerts low effort, $P(B|A_2)=$
- Event B^c: produce bad products
  - Given that the worker exerts high effort, $P(B^c|A_1)=$
  - Given that the worker exerts low effort, $P(B^c|A_2)=$

Bayes' theorem: a labor economics example



Bayes' theorem: a labor economics example

$$P(A_1 \mid B) = \frac{P(A_1 \text{ and } B)}{P(B)} = \frac{P(A_1 \text{ and } B)}{P(A_1 \text{ and } B) + P(A_2 \text{ and } B)}$$

$P(A_1|B)= 0.72/(0.72+0.06)=0.92$

- So, the manager knows that if a worker has produced good products, then the chance that this worker is a high effort worker is 92%.
- The manager has revised the prior probability of a high effort worker of 80% to a 92% posterior probability after observing good products.
- Would the manager give this worker a salary raise?