

ACE 261
Fall 2002
Prof. Katchova

Lecture 6

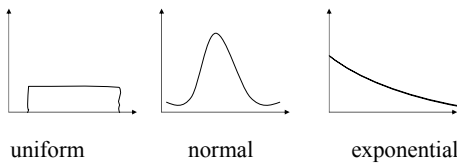
Continuous Probability Distributions

Outline

- Normal Probability Distribution (very, very important!)
- Uniform Probability Distribution
- Exponential Probability Distribution

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Continuous probability distributions



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Continuous probability distributions

- A continuous random variable can assume any value in an interval on the real line or in a collection of intervals.
- Continuous probability distributions are described by:
 - The interval of possible values for the variable x .
 - The probability density $f(x)$ associated with variable x .
- The probability density is not easily interpreted, but the area under the probability density is a probability.

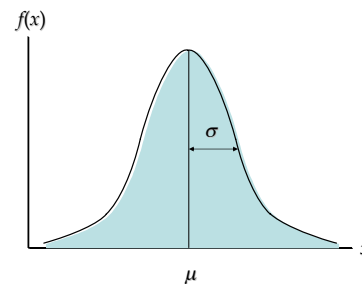
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Probability for a continuous distribution is the area under the curve!

- The probability of the random variable assuming a value within some given interval from x_1 to x_2 is defined to be the area under the graph of the probability density function between x_1 and x_2 .
- Total area under the probability density function (just as under the relative frequency histogram) = 1
- The probability of a random variable assuming a specific value is zero.

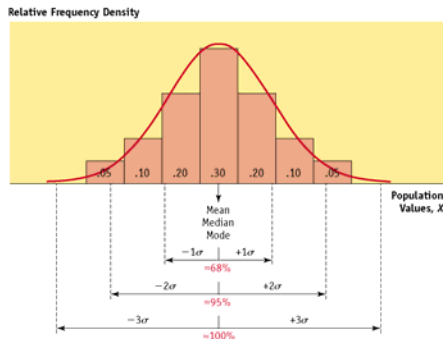
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Normal probability distribution graph



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Relative frequency on a histogram and probability density on a normal curve



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Characteristics of the normal probability distribution

- The shape of the normal curve is often illustrated as a bell-shaped curve.
- Two parameters, μ (mean) and σ (standard deviation), determine the location and shape of the distribution.
- The normal curve is symmetric. The highest point on the normal curve is at the mean, which is also the median and mode.
- The mean can be any numerical value: negative, zero, or positive.

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Characteristics of the normal probability distribution (continued)

- The standard deviation determines the width of the curve: larger values result in wider, flatter curves.
- The total area under the curve is 1 (.5 to the left of the mean and .5 to the right).
- Probabilities for the normal random variable are given by areas under the curve.

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The empirical rule for the normal probability distribution

- 68% of values of a normal random variable are within +/- 1 standard deviation of its mean.
- 95% of values of a normal random variable are within +/- 2 standard deviations of its mean.
- 99.7% of values of a normal random variable are within +/- 3 standard deviations of its mean.

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Normal probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where:

- μ = mean
- σ = standard deviation
- π = 3.14159
- e = 2.71828

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Standard normal probability distribution

- Standard normal probability distribution is a normal probability distribution with a mean of zero ($\mu=0$) and a standard deviation of one ($\sigma=1$).
- A standard normal random variable is usually denoted as z .

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The standard normal probability table

- It gives the probability (or the area under a curve) that the random variable z will be between the mean, $z=0$, and a specified value of z .
- The table is a little strange because it's a table of only one variable z but has both rows and columns. So the probability associated with $z=0.83$ is located at the 0.8 row and the 0.03 column.

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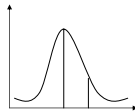
Using the standard normal probability table

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2549
.7	.2580	.2612	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389

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Using the standard normal probability tables

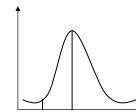
- What is the probability that a standard normal variable is between 0 and 1.26?
- This area is given in the tables
 $P(0 < z < 1.26) =$



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Using the standard normal probability tables

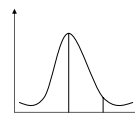
- What is the probability that a standard normal variable will be between -1.26 and 0?
- The trick is to notice that the normal distribution is symmetric.
 $P(-1.26 < z < 0) =$



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Using the standard normal probability tables

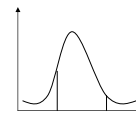
- What is the probability that a standard normal variable will be larger than 1.26?
- The trick is to notice that this is half of the area under the curve minus the area that we calculated before:
 $P(z > 1.26) = 0.5 - P(0 < z < 1.26) =$



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Using the standard normal probability tables

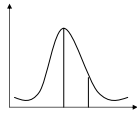
- What is the probability that a standard normal variable will be between -0.47 and 1.26?
- We break down the interval around zero.
 $P(-0.47 < z < 1.26) = P(-0.47 < z < 0) + P(0 < z < 1.26) =$



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Using the standard normal probability tables

- What is z value so that the probability of obtaining a larger value is 0.1?
- Notice that the area between 0 and z should be 0.4
- Look at the z tables and find the probability 0.4, the corresponding z value is 1.28.



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Find the following probabilities and z-values

- $P(0 < z < 2.45) =$
- $P(z > 2.45) =$
- $P(z > -0.68) =$
- $P(0.34 < z < 1.34) =$
- $P(0 < z < \underline{\quad}) = 0.34$
- $P(z > \underline{\quad}) = 0.55$

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Converting to the standard normal distribution

- But what if the variable is not standard normal? The statistical tables are only for standard normal variable z .
- Solution: convert the normal variable into a standard normal variable by changing each x value into a z value.
- Use the following formula. (It's just like the z-scores and is a measure of the number of standard deviations x is from μ).

$$z = (x - \mu) / \sigma$$

- Once the variable is converted to a standard normal variable, look at the standard normal tables for the probabilities.

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Normal probability distribution example

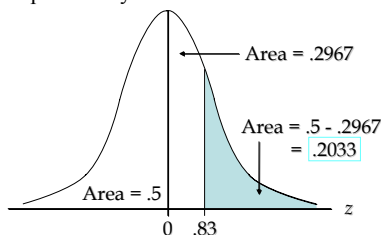
- An oil company sells on average 15 gallons of oil per week with a standard deviation of 6 gallons. What is the probability that the oil company will sell more than 20 gallons next week?
 $\mu =$
 $\sigma =$
- Before calculating probabilities we have to convert this normal distribution to a standard normal distribution.
 $z = (x - \mu) / \sigma =$

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Normal probability distribution example

$$P(x > 20) = P(z > 0.83) =$$

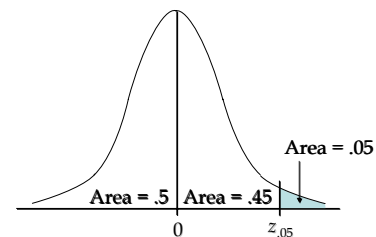
The oil company will sell more than 20 gallons with 20.33% probability.



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Normal probability distribution example

- If the oil company can keep in stock only limited number of gallons, how many gallons should it keep so that the probability of selling more than that is 0.05?
- The area between 0 and z must be 0.45. Find 0.45 as a probability and find the corresponding z value.



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Using the standard normal probability table

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767

$z_{.05} = 1.645$ is a reasonable estimate.
 This means the corresponding x value is 1.645 standard deviations above the mean.

Normal probability distribution example

- The corresponding value of x is given by

$$x = \mu + z_{.05}\sigma$$

$$= 15 + 1.645(6)$$

$$= 24.87$$
- The oil company should keep at least 24.87 gallons if it would like the probability of stock out to be 0.05.

Another example

- A grain elevator buys 100,000 bushels of corn per month with a standard deviation of 10,000 bushels.
- What is the probability that the grain elevator will buy between 90,000 and 120,000 bushels of corn next month?
- Usually, grain elevators re-sell the corn to processors. How many bushels is it safe to promise to sell next month if the elevator wants to be safe (i.e. have the promised bushels of corn) with 0.9 probability?

Uniform probability distribution

- A random variable is uniformly distributed whenever the probability is proportional to the interval's length.
 - The values of the x variable are limited in an interval.
- Examples of uniformly distributed variables that are equally likely to occur at any number within an interval.
 - Arrival time of bus, plane, or train within 20 minutes
 - Ripening of a crop within a month
 - Daily sales of a wholesaler
 - Rate of inflation next year
 - Time it takes to process a loan application

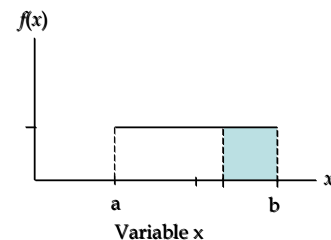
Uniform probability density function

$$f(x) = 1/(b - a) \text{ for } a \leq x \leq b$$

$$= 0 \text{ elsewhere}$$

where: a = smallest value the variable can assume
 b = largest value the variable can assume

Uniform probability distribution



Uniform probability distribution

- Expected value of x

$$E(x) = (a + b)/2$$
- Variance of x

$$\text{Var}(x) = (b - a)^2/12$$

where: a = smallest value the variable can assume
 b = largest value the variable can assume

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Uniform probability distribution example

- Slater customers are charged for the amount of salad they take. Sampling suggests that the amount of salad taken is uniformly distributed between 5 ounces and 15 ounces.

The probability density function is

$$f(x) = 1/10 \text{ for } 5 \leq x \leq 15$$

$$= 0 \text{ elsewhere}$$

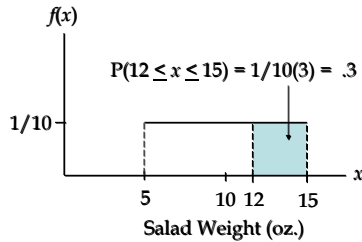
where:

x = salad plate filling weight

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Uniform probability distribution example

- What is the probability that a customer will take between 12 and 15 ounces of salad?



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Uniform probability distribution example

- Expected value of salad weight

$$E(x) = (a + b)/2$$

$$= (5 + 15)/2$$

$$= 10$$

- Variance of salad weight

$$\text{Var}(x) = (b - a)^2/12$$

$$= (15 - 5)^2/12$$

$$= 8.33$$

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Another uniform distribution example

- A TV show usually ends between 55 and 65 minutes after it started. What is the probability that it will end between 63 and 65 minutes after it started?

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Exponential probability distribution

- Exponential random variable describes the time or space between any two consecutive events in a Poisson process.
- Exponential probability distribution is used:
 - When there are only positive values of the x variable
 - When the smaller values of x are more likely than the larger values

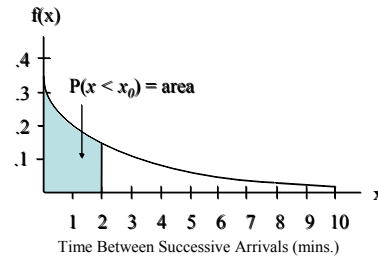
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Poisson and exponential distributions

- The Poisson distribution provides an appropriate description of the *number* of occurrences per interval
 - Number of flights arriving at an airport for a day
 - Number of students coming to ACE 261 today
 - Number of repairs needed on a mile of highway
- The exponential distribution provides an appropriate description of the *length of the interval* between occurrences
 - Minutes between flight arrivals
 - Time between students entering the classroom
 - Distance between each repair on the highway

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Graph of an exponential probability density function



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Exponential probability density function

$$f(x) = \frac{1}{\mu} e^{-x/\mu}$$

for $x \geq 0, \mu > 0$
 where μ = mean
 $e = 2.71828$

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Exponential probability distribution

- Cumulative exponential distribution function

$$P(x \leq x_0) = 1 - e^{-x_0/\mu}$$

where:

x_0 = some specific value of x

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Exponential probability distribution example

- The time between arrivals of cars at Al's Carwash follows an exponential probability distribution with a mean time between arrivals of 3 minutes. Al would like to know the probability that the time between two successive arrivals will be 2 minutes or less.
 $\mu=3$

$$P(x \leq 2) = 1 - 2.71828^{-2/3} = 1 - .5134 = .4866$$

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