ACE 261
Fall 2002
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Lecture 5

Discrete Probability Distributions

Outline

- Random variables
- Discrete probability distributions
- Expected value and variance
- Binomial probability distribution
- Poisson probability distribution
- Hypergeometric probability distribution

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Random variables

- A random variable is a numerical description of the outcome of an experiment.
- A random variable can be classified as being either discrete or continuous depending on the numerical values it assumes.

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Discrete random variables

- A discrete random variable may assume either a finite number of values or an infinite sequence of values.
  - Discrete random variable with a finite number of values
    x = number of TV sets sold at the store in one day
    x can take on 5 values (0, 1, 2, 3, 4)
  - Discrete random variable with an infinite sequence of values
    x = number of customers arriving in one week
    x can take on the values 0, 1, 2, . . .

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Continuous random variable

- A continuous random variable may assume any numerical value in an interval or collection of intervals.
  - Time, weight, distance, and temperature
  - x = time between TV sales in minutes
    x can take continuous values more than or equal to zero

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Discrete probability distributions

- The probability distribution for a random variable describes how probabilities are distributed over the values of the random variable.
- We can describe a discrete probability distribution with a table, graph, or equation.

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Discrete probability functions

- The probability distribution is defined by a probability function, denoted by $f(x)$, which provides the probability for each value of the random variable.
- The required conditions for a discrete probability function are:

$$f(x) \geq 0$$
$$\Sigma f(x) = 1$$

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Discrete probability distribution: tabular representation

Using past data on TV sales (below left), a tabular representation of the probability distribution for TV sales (below right) was developed.

Units Sold	Number of Days	x	$f(x)$
0	80	0	.40
1	50	1	.25
2	40	2	.20
3	10	3	.05
4	20	4	.10
	200		1.00

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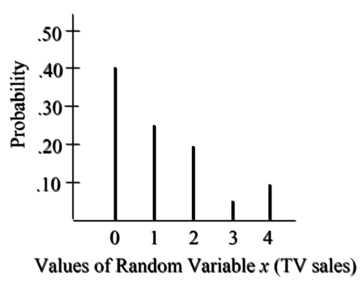
Discrete probability distribution: graphical representation



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Discrete uniform probability distribution

- The discrete uniform probability distribution is the simplest example of a discrete probability distribution given by a formula.
- The discrete uniform probability function is

$$f(x) = 1/n$$

where n = the number of values the random variable may assume.
- Note that the values of the random variable are equally likely.

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Expected Value and Variance

- The expected value, or mean, of a random variable is a measure of its central location.
  - Expected value of a discrete random variable:

$$E(x) = \mu = \Sigma x f(x)$$

- The variance summarizes the variability in the values of a random variable.
  - Variance of a discrete random variable:

$$\text{Var}(x) = \sigma^2 = \Sigma(x - \mu)^2 f(x)$$

- The standard deviation, σ, is defined as the positive square root of the variance.

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Expected value of a discrete random variable

x	$f(x)$	$xf(x)$
0	.40	.00
1	.25	.25
2	.20	.40
3	.05	.15
4	.10	.40
	$E(x)$ =	1.20

The expected number of TV sets sold in a day is 1.2

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Variance and standard deviation of a discrete random variable

x	$x - \mu$	$(x - \mu)^2$	$f(x)$	$(x - \mu)^2 f(x)$
0	-1.2	1.44	.40	.576
1	-0.2	0.04	.25	.010
2	0.8	0.64	.20	.128
3	1.8	3.24	.05	.162
4	2.8	7.84	.10	.784
				$1.660 = \sigma^2$

The variance of daily sales is 1.66 TV sets squared.

The standard deviation of sales is 1.2884 TV sets.

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Binomial probability distribution

- Properties of a Binomial Experiment
  - The experiment consists of a sequence of n identical trials.
  - Two outcomes, success and failure, are possible on each trial.
  - The probability of a success, denoted by p, does not change from trial to trial.
  - The trials are independent.

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Binomial probability distribution

- Binomial probability function

$$f(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{(n-x)}$$

where:

$f(x)$ = the probability of x successes in n trials

n = the number of trials

p = the probability of success on any one trial

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Binomial experiment: tossing a coin

- Toss a coin 5 times and record the number of heads.
  n=
  p=
- What is the probability of 3 heads?
  x =

$$f(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{(n-x)}$$

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Binomial experiment: tossing a coin binomial probability values from a table

x	$f(x)$
0	0.0312
1	0.1562
2	0.3125
3	0.3125
4	0.1562
5	0.0312

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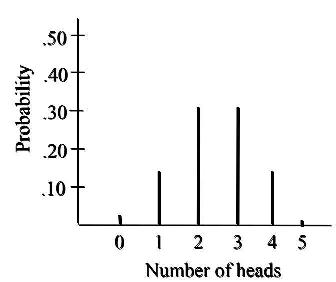
Binomial experiment: tossing a coin theoretical values



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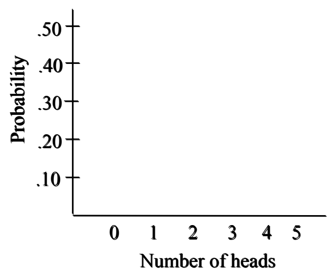
Binomial experiment: tossing a coin empirical values



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Binomial probability distribution example

- On the basis of past experience, management has seen a turnover of 10% of the hourly employees annually. Thus, for any hourly employees chosen at random, management estimates a probability of 0.1 that the person will not be with the company next year.
- Choosing 3 hourly employees a random, what is the probability that 1 of them will leave the company this year?
- $p =\quad , n = \quad , x =$

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Using the binomial probability function

$$f(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{(n-x)}$$

$$f(1) = \frac{3!}{1!(3-1)!} (0.1)^1 (0.9)^2$$

$=$

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Using the tables of binomial probabilities

		p								
n	**x**	.10	.15	.20	.25	.30	.35	.40	.45	.50
3	0	.7290	.6141	.5120	.4219	.3430	.2746	.2160	.1664	.1250
	1	.2430	.3251	.3840	.4219	.4410	.4436	.4320	.4084	.3750
	2	.0270	.0574	.0960	.1406	.1890	.2389	.2880	.3341	.3750
	3	.0010	.0034	.0080	.0156	.0270	.0429	.0640	.0911	.1250

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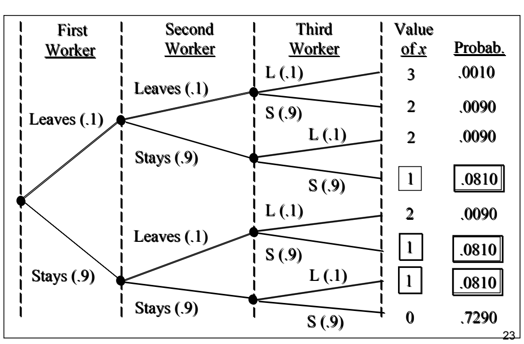
Using a tree diagram



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Binomial probability distribution

- Expected value

$$E(x) = \mu = np$$

- Variance

$$Var(x) = \sigma^2 = np(1-p)$$

- Standard deviation

$$\mathrm{SD}(x) = \sigma = \sqrt{np(1-p)}$$

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Binomial probability distribution example

- Expected Value
  $E(x) = \mu = 3(.1) = .3$ employees out of 3
- Variance
  $\text{Var}(x) = \sigma^2 = 3(.1)(.9) = .27$
- Standard Deviation
  $SD(x) = \sigma = \sqrt{3(.1)(.9)} = .52$ employees

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Poisson probability distribution

- Properties of a Poisson experiment
  - The probability of an occurrence is the same for any two intervals of equal length.
  - The occurrence or nonoccurrence in any interval is independent of the occurrence or nonoccurrence in any other interval.

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Poisson probability distribution

- Poisson probability function

$$f(x) = \frac{\mu^x e^{-\mu}}{x!}$$

where:

$f(x)$ = probability of x occurrences in an interval

μ = mean number of occurrences in an interval

$e = 2.71828$

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Poisson probability distribution

- To solve a problem, you must figure out what are

  μ = mean number of occurrences in an interval
  - The time or space intervals must match with the interval that we are interested in.

  x = the value of the discrete random variable that we are interested in.

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Poisson probability distribution example

- Patients arrive at the emergency room of Carl Clinic at the average rate of 6 per hour on weekend evenings. What is the probability of 4 arrivals in 30 minutes on a weekend evening?

$$\mu = 6/\text{hour} = 3/\text{half hour}, \quad x = 4$$

$$f(4) = \frac{3^4(2.71828)^{-3}}{\;} = .1680$$

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Using the tables of Poisson probabilities

	μ									
x	**2.1**	**2.2**	**2.3**	**2.4**	**2.5**	**2.6**	**2.7**	**2.8**	**2.9**	**3.0**
0	.1225	.1108	.1003	.0907	.0821	.0743	.0672	.0608	.0550	.0498
1	.2572	.2438	.2306	.2177	.2052	.1931	.1815	.1703	.1596	.1494
2	.2700	.2681	.2652	.2613	.2565	.2510	.2450	.2384	.2314	.2240
3	.1890	.1966	.2033	.2090	.2138	.2176	.2205	.2225	.2237	.2240
4	.0992	.1082	.1169	.1254	.1336	.1414	.1488	.1557	.1622	.1680
5	.0417	.0476	.0538	.0602	..0668	.0735	.0804	.0872	.0940	.1008
6	.0146	.0174	.0206	.0241	.0278	.0319	.0362	.0407	.0455	.0504
7	.0044	.0055	.0068	.0083	.0099	.0118	.0139	.0163	.0188	.0216
8	.0011	.0015	.0019	.0025	.0031	.0038	.0047	.0057	.0068	.0081

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Hypergeometric Probability Distribution

- The hypergeometric distribution is closely related to the binomial distribution.
- With the hypergeometric distribution, the trials are not independent, and the probability of success changes from trial to trial.

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Hypergeometric probability distribution

- Hypergeometric Probability Function

$$f(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}} \qquad \text{for } 0 \le x \le r$$

where: $f(x)$ = probability of x successes in n trials
n = number of trials
N = number of elements in the population
r = number of elements in the population labeled success

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Hypergeometric probability distribution example

- Bob has removed two dead batteries from a flashlight and inadvertently mingled them with the two good batteries he intended as replacements. The four batteries look identical.
- Bob now randomly selects two of the four batteries. What is the probability he selects the two good batteries?

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Hypergeometric probability distribution example

- Probability that the first battery will be good is = 2/4 = ½
- Given that the first battery was good, the probability that the second battery will be good is =1/3
- Probability that both batteries will be good =(1/2)*(1/3)=1/6

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Hypergeometric probability distribution example

$$f(x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}} = \frac{\binom{2}{2}\binom{2}{0}}{\binom{4}{2}} = \frac{\left(\frac{2!}{2!0!}\right)\left(\frac{2!}{0!2!}\right)}{\left(\frac{4!}{2!2!}\right)} = \frac{1}{6} = .167$$

where:
x = 2 = number of good batteries selected
r = 2 = number of good batteries in total
n = 2 = number of batteries selected
N = 4 = number of batteries in total

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