Chapter 3
Mathematics in daily life
Part 2

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Permutations and Combinations

Permutations vs. Combinations

- Both are ways to count the possibilities
- The difference between them is whether order matters or not
- Consider a poker hand:
  - A♦, 5♥, 7♣, 10♠, K♠
- Is that the same hand as:
  - K♠, 10♠, 7♣, 5♥, A♦
- Does the order the cards are handed out matter?
  - If yes, then we are dealing with permutations
  - If no, then we are dealing with combinations

Permutations

- A permutation is an ordered arrangement of the elements of some set *S*
  - Let *S* = {a, b, c}
  - c, b, a is a permutation of S
  - b, c, a is a *different* permutation of S
- An *r*-permutation is an ordered arrangement of *r* elements of the set
  - A♦, 5♥, 7♣, 10♠, K♠ is a 5-permutation of the set of cards
- The notation for the number of *r*-permutations: $P(n,r)$
  - The poker hand is one of P(52,5) permutations

Permutations

- Number of poker hands (5 cards):
  - *P*(52,5) = 52*51*50*49*48 = 311,875,200
- Number of (initial) blackjack hands (2 cards):
  - *P*(52,2) = 52*51 = 2,652
- *r*-permutation notation: *P*(*n*,*r*)
  - The poker hand is one of P(52,5) permutations

$$P(n,r) = n(n-1)(n-2)...(n-r+1)$$

$$= \frac{n!}{(n-r)!}$$

$$= \prod_{i=n-r+1}^{n} i$$

r-permutations example

- How many ways are there for 5 people in this class to give presentations?

- There are 27 students in the class
  - P(27,5) = 27*26*25*24*23 = 9,687,600
  - Note that the order they go in does matter in this example!

Permutations vs. *r*-permutations

- *r*-permutations: Choosing an ordered 5 card hand is *P*(52,5)
  - When people say "permutations", they almost always mean *r*-permutations
    - But the name can refer to both

- Permutations: Choosing an order for all 52 cards is *P*(52,52) = 52!
  - Thus, *P*(*n*,*n*) = *n*!

Rosen, section 4.3, question 3

- How many permutations of {a, b, c, d, e, f, g} end with a?
  - Note that the set has 7 elements

- The last character must be a
  - The rest can be in any order
- Thus, we want a 6-permutation on the set {b, c, d, e, f, g}
- P(6,6) = 6! = 720

- Why is it not P(7,6)?

Combinations

- What if order *doesn't* matter?
- In poker, the following two hands are equivalent:
  - A♦, 5♥, 7♣, 10♠, K♠
  - K♠, 10♠, 7♣, 5♥, A♦

- The number of *r*-combinations of a set with *n* elements, where *n* is non-negative and 0≤*r*≤*n* is:

$$C(n,r)=\frac{n!}{r!(n-r)!}$$

Combinations example

- How many different poker hands are there (5 cards)?

$$C(52,5)=\frac{52!}{5!(52-5)!}=\frac{52!}{5!47!}=\frac{52*51*50*49*48*47!}{5*4*3*2*1*47!}=2{,}598{,}960$$

- How many different (initial) blackjack hands are there?

$$C(52,2)=\frac{52!}{2!(52-2)!}=\frac{52!}{2!50!}=\frac{52*51}{2*1}=1{,}326$$

Combination formula proof

- Let $C(52,5)$ be the number of ways to generate unordered poker hands
- The number of ordered poker hands is $P(52,5)$ = 311,875,200
- The number of ways to order a single poker hand is $P(5,5) = 5! = 120$
- The total number of unordered poker hands is the total number of ordered hands divided by the number of ways to order each hand
- Thus, $C(52,5) = P(52,5)/P(5,5)$

Combination formula proof

- Let $C(n,r)$ be the number of ways to generate unordered combinations
- The number of ordered combinations (i.e. *r*-permutations) is $P(n,r)$
- The number of ways to order a single one of those *r*-permutations $P(r,r)$
- The total number of unordered combinations is the total number of ordered combinations (i.e. *r*-permutations) divided by the number of ways to order each combination
- Thus, $C(n,r) = P(n,r)/P(r,r)$

Combination formula proof

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!(n-r)!}$$

- Note that the textbook explains it slightly differently, but it is same proof

Rosen, section 4.3, question 11

- How many bit strings of length 10 contain:

c) at least four 1's?
  - There can be 4, 5, 6, 7, 8, 9, or 10 occurrences of 1
  - Thus, the answer is:
    - $C(10,4) + C(10,5) + C(10,6) + C(10,7) + C(10,8) + C(10,9) + C(10,10)$
    - = 210+252+210+120+45+10+1
    - = 848
  - Alternative answer: subtract from 2^{10} the number of strings with 0, 1, 2, or 3 occurrences of 1

d) an equal number of 1's and 0's?
  - Thus, there must be five 0's and five 1's
  - Find the positions of the five 1's
  - Thus, the answer is $C(10,5) = 252$

Corollary example

- There are C(52,5) ways to pick a 5-card poker hand
- There are C(52,47) ways to pick a 47-card hand
- P(52,5) = 2,598,960 = P(52,47)

- When dealing 47 cards, you are picking 5 cards to not deal
  - As opposed to picking 5 card to deal
  - Again, the order the cards are dealt in does matter

Rosen, section 4.3, question 42

- How many ways are there for 4 horses to finish if ties are allowed?
  - Note that order does matter!
- Solution by cases
  - No ties
    - The number of permutations is P(4,4) = 4! = 24
  - Two horses tie
    - There are *C*(4,2) = 6 ways to choose the two horses that tie
    - There are *P*(3,3) = 6 ways for the "groups" to finish
      - A "group" is either a single horse or the two tying horses
    - By the product rule, there are 6*6 = 36 possibilities for this case
  - Two groups of two horses tie
    - There are *C*(4,2) = 6 ways to choose the two winning horses
    - The other two horses tie for second place
  - Three horses tie with each other
    - There are *C*(4,3) = 4 ways to choose the two horses that tie
    - There are *P*(2,2) = 2 ways for the "groups" to finish
    - By the product rule, there are 4*2 = 8 possibilities for this case
  - All four horses tie
    - There is only one combination for this
  - By the sum rule, the total is 24+36+6+8+1 = 75

A last note on combinations

- An alternative (and more common) way to denote an *r*-combination:

$$C(n,r) = \binom{n}{r}$$

- I'll use C(*n*,*r*) whenever possible, as it is easier to write in PowerPoint

reference

Special thank:

CS/APMA 102 Rosen Section 4.3

- Aaron Bloomfield