1. Write the upper triangular matrix $A$ of order 4, given that all entries which are not required to be 0 are equal to the sum of their subscripts. (For example, $A_{23} = 2 + 3 = 5$.)

Solution: We have $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 4 & 5 & 6 \\ 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix}$.

2. (a) Construct the matrix $A = [A_{ij}]$ if $A$ is $2 \times 3$ and $A_{ij} = -i + 2j$.
(b) Construct the $2 \times 4$ matrix $C = [(i + j)^2]$.

Solution: (a) We have

$$A = \begin{bmatrix} -1 + 2(1) & -1 + 2(2) & -1 + 2(3) \\ -2 + 2(1) & -2 + 2(2) & -2 + 2(3) \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \end{bmatrix}.$$

(b) We have

$$C = \begin{bmatrix} (1 + 1)^2 & (1 + 2)^2 & (1 + 3)^2 & (1 + 4)^2 \\ (2 + 1)^2 & (2 + 2)^2 & (2 + 3)^2 & (2 + 4)^2 \end{bmatrix} = \begin{bmatrix} 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \end{bmatrix}.$$

3. (a) Construct the matrix $B = [B_{ij}]$ if $B$ is $2 \times 2$ and $B_{ij} = (-1)^{i-j}(i^2 - j^2)$.
(b) Construct the $2 \times 3$ matrix $D = [(−1)^i(j^3)]$.

Solution: (a) We have

$$B = \begin{bmatrix} (-1)^{1-1}(1^2 - 1^2) & (-1)^{1-2}(1^2 - 2^2) \\ (-1)^{2-1}(2^2 - 1^2) & (-1)^{2-2}(2^2 - 2^2) \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}.$$
(b) We have

\[
D = \begin{bmatrix}
(-1)^1(1^3) & (-1)^1(2^3) & (-1)^1(3^3) \\
(-1)^2(1^3) & (-1)^2(2^3) & (-1)^2(3^3)
\end{bmatrix} = \begin{bmatrix} -1 & -8 & -27 \\
1 & 8 & 27 \end{bmatrix}.
\]

4. If \( A = [A_{ij}] \) is \( 12 \times 10 \), how many entries does \( A \) have? If \( A_{ij} = 1 \) for \( i = j \) and \( A_{ij} = 0 \) for \( i \neq j \), find \( A_{33}, A_{52}, A_{10,10}, \) and \( A_{12,10} \).

**Solution:** Since \( 12 \cdot 10 = 120 \), the matrix \( A \) has 120 entries. For \( A_{33} \), we have \( i = 3 = j \), so \( A_{33} = 1 \). Since \( 5 \neq 2 \), \( A_{52} = 0 \). For \( A_{10,10} \), we have \( i = 10 = j \), so \( A_{10,10} = 1 \). Since \( 12 \neq 10 \), \( A_{12,10} = 0 \).

5. A matrix is **symmetric** if \( A^T = A \). Is the following matrix symmetric?

(a) \( A = \begin{bmatrix} 2 & 5 & -3 & 0 \\
0 & 3 & 6 & 2 \\
7 & 8 & -2 & 1 \end{bmatrix} \)

(b) \( A = \begin{bmatrix} 2 & -1 & 0 \\
-1 & 5 & 1 \\
0 & 1 & 3 \end{bmatrix} \)

**Solution:** (a) We have \( A^T = \begin{bmatrix} 2 & 5 & -3 & 0 \\
0 & 3 & 6 & 2 \\
7 & 8 & -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & 7 \\
5 & 3 & 8 \\
-3 & 6 & -2 \\
0 & 2 & 1 \end{bmatrix} \). We see that \( A^T \neq A \).

So, the matrix \( A \) is not symmetric.

(b) We have \( A^T = \begin{bmatrix} 2 & -1 & 0 \\
-1 & 5 & 1 \\
0 & 1 & 3 \end{bmatrix}^T = \begin{bmatrix} 2 & -1 & 0 \\
-1 & 5 & 1 \\
0 & 1 & 3 \end{bmatrix} \). We see that \( A^T = A \). So, the matrix \( A \) is symmetric.

6. A grocery sold 125 cans of tomato soup, 275 cans of beans, and 400 cans of tuna. Write a row vector that gives the number of each item sold. If the items sell for $0.95, $1.03, and $1.25 each, respectively, write this information as a column vector.

**Solution:** We have row vector = \( \begin{bmatrix} 125 & 275 & 400 \end{bmatrix} \) and column vector = \( \begin{bmatrix} 0.95 \\
1.03 \\
1.25 \end{bmatrix} \).
7. The Widget Company has its monthly sales reports given by means of matrices whose rows, in order, represent the number of regular, deluxe, and extreme models sold, and the columns, in order, give the number of red, white, blue, and purple units sold. The matrices for January and February are

\[
J = \begin{bmatrix}
1 & 4 & 5 & 0 \\
3 & 5 & 2 & 7 \\
4 & 1 & 3 & 2 \\
\end{bmatrix} \quad F = \begin{bmatrix}
2 & 5 & 7 & 7 \\
2 & 4 & 4 & 6 \\
0 & 0 & 1 & 2 \\
\end{bmatrix}
\]

respectively. (a) How many white extreme models were sold in January?  (b) How many blue deluxe models were sold in February? (c) In which month were more purple regular models sold?  (d) Which models and which colors sold the same number of units in both months?  (e) In which month were more deluxe models sold?  (f) In which month were more red widgets sold?  (g) How many widgets were sold in January?

**Solution:**

(a) From \( J \), the entry in row 3 (extreme) and column 2 (white) is 1. Thus in January, 1 white extreme model was sold.

(b) From \( F \), the entry in row 2 (deluxe) and column 3 (blue) is 4. Thus in February, 4 blue deluxe models were sold.

(c) The entries in row 1 (regular) and column 4 (purple) give the number of purple regular models sold. For \( J \) the entry is 0 and for \( F \) the entry is 7. Thus more purple regular models were sold in February.

(d) Only the entries in row 3 (extreme) and column 4 (purple) of matrices \( J \) and \( F \) are the same. Thus purple extreme models were sold the same number of units in both months.

(e) In January a total of \( 3 + 5 + 2 = 7 = 17 \) deluxe models were sold. In February a total of \( 2 + 4 + 4 + 6 = 16 \) deluxe models were sold. Thus, more deluxe models were sold in January.

(f) In January a total of \( 1 + 3 + 4 = 8 \) red widgets were sold, while in February a total of \( 2 + 2 + 0 = 4 \) red widgets were sold. Thus, more red widgets were sold in January.

(g) Adding all entries in matrix \( J \) yields that a total of 37 widgets were sold in January.

8. An auto parts company manufactures distributors, sparkplugs, and magnetos at two plants, I and II. Matrix \( X \) represents the production of the two plants for retailer X, and matrix \( Y \) represents the production of the two plants for retailer Y. Write a matrix that represents the total production at the two plants for both retailers, where
\[ X = \begin{bmatrix} \text{DIS} \\ \text{SPG} \\ \text{MAG} \end{bmatrix} \begin{bmatrix} 30 & 50 \\ 800 & 720 \\ 25 & 30 \end{bmatrix}, \quad Y = \begin{bmatrix} \text{DIS} \\ \text{SPG} \\ \text{MAG} \end{bmatrix} \begin{bmatrix} 15 & 25 \\ 960 & 800 \\ 10 & 5 \end{bmatrix} \]

**Solution:** We find
\[
X + Y = \begin{bmatrix} 30 & 50 \\ 800 & 720 \\ 25 & 30 \end{bmatrix} + \begin{bmatrix} 15 & 25 \\ 960 & 800 \\ 10 & 5 \end{bmatrix} = \begin{bmatrix} 30 + 15 & 50 + 25 \\ 800 + 960 & 720 + 800 \\ 25 + 10 & 30 + 5 \end{bmatrix} = \begin{bmatrix} 45 & 75 \\ 1760 & 1520 \\ 35 & 35 \end{bmatrix}.
\]

9. Let matrix \( A \) represent the sales (in thousands of dollars) of a toy company in 2007 in three cities, and let \( B \) represent the sales in the same cities in 2009, where
\[
A = \begin{bmatrix} \text{Action} \\ \text{Educational} \end{bmatrix} \begin{bmatrix} 400 & 350 & 150 \\ 450 & 280 & 850 \end{bmatrix},
\]
\[
B = \begin{bmatrix} \text{Action} \\ \text{Educational} \end{bmatrix} \begin{bmatrix} 380 & 330 & 220 \\ 460 & 320 & 750 \end{bmatrix}
\]

If the company buys a competitor and doubles its 2009 sales in 2010, what is the change in sales between 2007 and 2010?

**Solution:** We find
\[
2B - A = 2 \begin{bmatrix} 380 & 330 & 220 \\ 460 & 320 & 750 \end{bmatrix} - \begin{bmatrix} 400 & 350 & 150 \\ 450 & 280 & 850 \end{bmatrix},
\]
\[
= \begin{bmatrix} 2 \cdot 380 & 2 \cdot 330 & 2 \cdot 220 \\ 2 \cdot 460 & 2 \cdot 320 & 2 \cdot 750 \end{bmatrix} - \begin{bmatrix} 400 & 350 & 150 \\ 450 & 280 & 850 \end{bmatrix},
\]
\[
= \begin{bmatrix} 760 & 660 & 440 \\ 920 & 640 & 1500 \end{bmatrix} - \begin{bmatrix} 400 & 350 & 150 \\ 450 & 280 & 850 \end{bmatrix},
\]
\[
= \begin{bmatrix} 360 & 310 & 290 \\ 470 & 360 & 650 \end{bmatrix}.
\]

10. Suppose the prices of products A, B, C, and D are given, in that order, by the price row vector
\[
P = \begin{bmatrix} p_A \\ p_B \\ p_C \\ p_D \end{bmatrix}.
\]

If the prices are to increased by 16\%, the vector for the new prices can be obtained by multiplying \( P \) by what scalar?
Solution: We find

\[ P + 0.16P = \begin{bmatrix} p_A & p_B & p_C & p_D \end{bmatrix} + 0.16 \begin{bmatrix} p_A & p_B & p_C & p_D \end{bmatrix} = \begin{bmatrix} 1.16p_A & 1.16p_B & 1.16p_C & 1.16p_D \end{bmatrix} = 1.16P. \]

Thus \( P \) must be multiplied by 1.16.

11. A pet store has 6 kittens, 10 puppies, and 7 parrots in stock. If the value of each kitten is $55, each puppy is $150, and each parrot is $35, find the total value of the pet store’s inventory using matrix multiplication.

Solution: We find

\[
\begin{bmatrix} 6 & 10 & 7 \\ 150 \end{bmatrix} = 6 \cdot 55 + 10 \cdot 150 + 7 \cdot 35 = 330 + 1500 + 245 = 2075.
\]

Hence the total value of the pet store’s inventory is $2075.

12. Suppose that a building contractor has accepted orders for five ranch-style houses, seven Cape Cod-style houses, and 12 colonial-style houses. Furthermore, suppose that the “raw materials” that go into each type of house are steel, wood, glass, paint, and labor. The entries in the following matrix \( R \) give the number of units of each raw material going into each type of house:

\[
R = \begin{bmatrix}
5 & 20 & 16 & 7 & 17 \\
7 & 18 & 12 & 9 & 21 \\
6 & 25 & 8 & 5 & 13
\end{bmatrix}
\]

Suppose steel costs $2500 per unit, wood costs $1200 per unit, and glass, paint, and labor cost $800, $150, and $1500 per unit, respectively. Using matrix multiplication, compute the total cost of raw materials.

Solution: Represent the orders of the houses by the row vector \( Q = \begin{bmatrix} 5 & 7 & 12 \end{bmatrix} \). Then the amount of each raw material needed to fulfill the orders is given by the matrix

\[
QR = \begin{bmatrix} 5 & 7 & 12 \end{bmatrix} \begin{bmatrix}
5 & 20 & 16 & 7 & 17 \\
7 & 18 & 12 & 9 & 21 \\
6 & 25 & 8 & 5 & 13
\end{bmatrix} = \begin{bmatrix} 146 & 526 & 260 & 158 & 388 \end{bmatrix}.
\]
Thus, the contractor should order 146 units of steel, 526 units of wood, 260 units of glass, and so on. Represent the costs of the materials by the column vector

$$C = \begin{bmatrix} 2500 \\ 1200 \\ 800 \\ 150 \\ 1500 \end{bmatrix}.$$ 

Then the total cost of raw materials for all the houses is given by

$$QRC = (QR)C = \begin{bmatrix} 2500 \\ 1200 \\ 800 \\ 150 \\ 1500 \end{bmatrix} = \begin{bmatrix} 1,809,900 \end{bmatrix}.$$ 

The total cost is $1,809,900.

13. In Problem 12, assume that the contractor wishes to take into account the cost of transporting raw materials to the building site, as well as the purchasing cost. Suppose the costs are given in the following matrix:

$$C = \begin{bmatrix} 3500 & 50 \\ 1500 & 50 \\ 1000 & 100 \\ 250 & 10 \\ 3500 & 0 \end{bmatrix}$$

(a) Find a matrix whose entries give the purchase and transportation costs of the materials for each type of the house.

(b) Find the matrix whose first entry gives the total purchase price and whose second entry gives the total transportation cost.

(c) Using matrix multiplication, find the total cost of materials and transportation for all houses being built.

**Solution:**

(a) The matrix is given by

$$RC = \begin{bmatrix} 5 & 20 & 16 & 7 & 17 \\ 7 & 18 & 12 & 9 & 21 \\ 6 & 25 & 8 & 5 & 13 \\ 3500 & 50 \\ 1500 & 50 \\ 1000 & 100 \\ 250 & 10 \\ 3500 & 0 \end{bmatrix} = \begin{bmatrix} 124,750 & 2920 \\ 139,250 & 2540 \\ 113,250 & 2400 \end{bmatrix}.$$
(b) The matrix is given by

\[
QRC = Q(RC) = \begin{bmatrix}
5 & 7 & 12
\end{bmatrix}
\begin{bmatrix}
124,750 & 2920 \\
139,250 & 2540 \\
113,250 & 2400
\end{bmatrix}
= \begin{bmatrix}
2,957,500 & 61,180
\end{bmatrix}.
\]

(c) Let \( Z = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \). Then the total transportation cost is given by

\[
QRCZ = (QRC)Z = \begin{bmatrix}
2,957,500 & 61,180
\end{bmatrix}
\begin{bmatrix}
1 \\ 1
\end{bmatrix}
= \begin{bmatrix}
3,018,680
\end{bmatrix}.
\]