Notes on ICNS 100

Chapter 6: 6.6 Inverses; Special Topics: Determinant, Cramer's Rule

1. If A is a square matrix and there exists a matrix C such that $CA = I$, then C is called an <u>inverse</u> of A, and A is said to be <u>invertible</u>. We denote the inverse of A if it exists by A^{-1}.

2. If A is an invertible matrix, then the matrix equation $AX = B$ has the unique solution $X = A^{-1}B$.

3. <u>Theorem</u> Let A be a square matrix and let A_k be the reduced matrix obtained from A by elementary row operations. Then A is invertible if and only if $A_k = I$. Moreover, if $E_1, E_2, \ldots, E_k$ is a sequence of elementary row operations that takes A to I, then the same sequence takes I to A^{-1}.

   <u>Example</u> Use elemetary row operations to find inverse, if any, of the following matrices.

$$\begin{bmatrix} 6 & 1 \\ 7 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 5 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{bmatrix}$$

4. (a) A <u>submatrix</u> of an $m \times n$ matrix A is a matrix obtained from A by removing one or more of its rows and/or one or more of its columns.

   (b) If A is an $n \times n$ matrix, we shall use the notation A_{rs} to denote the submatrix of A obtained by deleting the rth row and sth column of A.

5. The <u>determinant</u> of the 1×1 and 2×2 matrices are defined as follows.

$$\det[a] = a \qquad \text{and} \qquad \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

6. For any $n \times n$ matrix with $n \geq 2$, the <u>minor</u> belonging to the (i, j) entry a_{ij} of A is $M_{ij} = \det A_{ij}$. The <u>cofactor</u> of a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$.

7. For $n \geq 3$ the determinant of $n \times n$ matrix A is given by

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

   for any fixed i, $i = 1, 2, \ldots, n$ (expansion along the ith row of A.)

   or

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

   for any fixed j, $j = 1, 2, \ldots, n$ (expansion along the jth column of A.)

8. <u>Theorem</u> The square matrix is invertible if and only if its determinant is nonzero.

<u>Example</u> Find determinant of the following matrices. Determine which matrices is invertible.

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 & 4 & -2 \\ 4 & 3 & 0 & -1 \\ 2 & 4 & 1 & 3 \\ -2 & 1 & 0 & 4 \end{bmatrix}$$

9. Cramer's Rule

If the coefficient matrix A of the system of n linear equations and n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

is invertible, then the system has a unique solution given by

$$x_j = \frac{\det B_j}{\det A}, \quad j = 1, 2, \ldots, n$$

where B_j is the $n \times n$ matrix obtained from A by replacing the jth column of A with the column of constants on the right side of the equations.

<u>Example</u> Use Cramer's rule to solve the following systems.

(a) $\begin{cases} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 + x_3 &= 2 \\ -x_1 + x_2 + 2x_3 &= 4 \end{cases}$ (b) $\begin{cases} x_1 + 2x_2 + 3x_3 &= -1 \\ 2x_1 + x_2 - 4x_3 &= 9 \\ x_1 - x_2 + 2x_3 &= -2 \end{cases}$

(c) $\begin{cases} x_1 + x_3 - x_4 &= -4 \\ 2x_1 + x_2 - x_3 + x_4 &= 8 \\ -x_1 + 2x_2 - 2x_4 &= -5 \\ x_1 + 2x_3 + 2x_4 &= 3 \end{cases}$

<u>Assignment</u> Do Problems 6.6: 2, 4, 6, 12, 18, 24, 32, 34, 36.