Notes on ICNS 100

Chapter 6: 6.6 Inverses; Special Topics: Determinant, Cramer's Rule

1. If A is a square matrix and there exists a matrix C such that $CA = I$, then C is called an inverse of A, and A is said to be invertible. We denote the inverse of A if it exists by A^{-1}.

2. If A is an invertible matrix, then the matrix equation $AX = B$ has the unique solution $X = A^{-1}B$.

3. Theorem Let A be a square matrix and let A_k be the reduced matrix obtained from A by elementary row operations. Then A is invertible if and only if $A_k = I$. Moreover, if $E_1, E_2, \ldots, E_k$ is a sequence of elementary row operations that takes A to I, then the same sequence takes I to A^{-1}.

   Example Use elemetary row operations to find inverse, if any, of the following matrices.

$$\begin{bmatrix} 6 & 1 \\ 7 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 5 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{bmatrix}$$

4. (a) A submatrix of an $m \times n$ matrix A is a matrix obtained from A by removing one or more of its rows and/or one or more of its columns.

   (b) If A is an $n \times n$ matrix, we shall use the notation A_{rs} to denote the submatrix of A obtained by deleting the rth row and sth column of A.

5. The determinant of the 1×1 and 2×2 matrices are defined as follows.

$$\det[a] = a \qquad \text{and} \qquad \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

6. For any $n \times n$ matrix with $n \geq 2$, the minor belonging to the (i, j) entry a_{ij} of A is $M_{ij} = \det A_{ij}$. The cofactor of a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$.

7. For $n \geq 3$ the determinant of $n \times n$ matrix A is given by

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

   for any fixed i, $i = 1, 2, \ldots, n$ (expansion along the ith row of A.)

   or

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

   for any fixed j, $j = 1, 2, \ldots, n$ (expansion along the jth column of A.)

8. Theorem The square matrix is invertible if and only if its determinant is nonzero.

Example Find determinant of the following matrices. Determine which matrices is invertible.

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 & 4 & -2 \\ 4 & 3 & 0 & -1 \\ 2 & 4 & 1 & 3 \\ -2 & 1 & 0 & 4 \end{bmatrix}$$

9. Cramer's Rule

If the coefficient matrix A of the system of n linear equations and n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots& \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

is invertible, then the system has a unique solution given by

$$x_j = \frac{\det B_j}{\det A}, \quad j = 1, 2, \ldots, n$$

where B_j is the $n \times n$ matrix obtained from A by replacing the jth column of A with the column of constants on the right side of the equations.

Example Use Cramer's rule to solve the following systems.

$$\text{(a)} \begin{cases} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 + x_3 &= 2 \\ -x_1 + x_2 + 2x_3 &= 4 \end{cases} \qquad \text{(b)} \begin{cases} x_1 + 2x_2 + 3x_3 &= -1 \\ 2x_1 + x_2 - 4x_3 &= 9 \\ x_1 - x_2 + 2x_3 &= -2 \end{cases}$$

$$\text{(c)} \begin{cases} x_1 + x_3 - x_4 &= -4 \\ 2x_1 + x_2 - x_3 + x_4 &= 8 \\ -x_1 + 2x_2 - 2x_4 &= -5 \\ x_1 + 2x_3 + 2x_4 &= 3 \end{cases}$$

Assignment Do Problems 6.6: 2, 4, 6, 12, 18, 24, 32, 34, 36.