Chapter 6: 6.6 Inverses; Special Topics: Determinant, Cramer’s Rule

1. If \( A \) is a square matrix and there exists a matrix \( C \) such that \( CA = I \), then \( C \) is called an inverse of \( A \), and \( A \) is said to be invertible. We denote the inverse of \( A \) if it exists by \( A^{-1} \).

2. If \( A \) is an invertible matrix, then the matrix equation \( AX = B \) has the unique solution \( X = A^{-1}B \).

3. Theorem Let \( A \) be a square matrix and let \( A_k \) be the reduced matrix obtained from \( A \) by elementary row operations. Then \( A \) is invertible if and only if \( A_k = I \). Moreover, if \( E_1, E_2, ..., E_k \) is a sequence of elementary row operations that takes \( A \) to \( I \), then the same sequence takes \( I \) to \( A^{-1} \).

Example Use elementary row operations to find inverse, if any, of the following matrices.

\[
\begin{bmatrix}
6 & 1 \\
7 & 1 \\
\end{bmatrix},
\begin{bmatrix}
1 & 2 & 3 \\
0 & 0 & 4 \\
0 & 0 & 5 \\
\end{bmatrix},
\begin{bmatrix}
2 & 3 & -1 \\
1 & 2 & 1 \\
-1 & -1 & 3 \\
\end{bmatrix}
\]

4. (a) A submatrix of an \( m \times n \) matrix \( A \) is a matrix obtained from \( A \) by removing one or more of its rows and/or one or more of its columns.

(b) If \( A \) is an \( n \times n \) matrix, we shall use the notation \( A_{rs} \) to denote the submatrix of \( A \) obtained by deleting the \( r \)th row and \( s \)th column of \( A \).

5. The determinant of the \( 1 \times 1 \) and \( 2 \times 2 \) matrices are defined as follows.

\[
det[a] = a \quad \text{and} \quad det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.
\]

6. For any \( n \times n \) matrix with \( n \geq 2 \), the minor belonging to the \((i, j)\) entry \( a_{ij} \) of \( A \) is \( M_{ij} = \det A_{ij} \). The cofactor of \( a_{ij} \) is \( C_{ij} = (-1)^{i+j}M_{ij} \).

7. For \( n \geq 3 \) the determinant of \( n \times n \) matrix \( A \) is given by

\[
det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{n1}C_{n1}
\]

for any fixed \( i, i = 1, 2, \ldots, n \) (expansion along the \( i \)th row of \( A \))

or

\[
det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}
\]

for any fixed \( j, j = 1, 2, \ldots, n \) (expansion along the \( j \)th column of \( A \)).
8. **Theorem** The square matrix is invertible if and only if its determinant is nonzero.

Example Find determinant of the following matrices. Determine which matrices is invertible.

\[
\begin{bmatrix}
1 & 2 & -1 \\
0 & 3 & 1 \\
2 & 1 & 4 \\
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 3 & 4 & -2 \\
4 & 3 & 0 & -1 \\
2 & 4 & 1 & 3 \\
-2 & 1 & 0 & 4 \\
\end{bmatrix}
\]

9. **Cramer’s Rule**

If the coefficient matrix \( A \) of the system of \( n \) linear equations and \( n \) unknowns

\[
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\vdots \quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n
\]

is invertible, then the system has a unique solution given by

\[
x_j = \frac{\det B_j}{\det A}, \quad j = 1, 2, \ldots, n
\]

where \( B_j \) is the \( n \times n \) matrix obtained from \( A \) by replacing the \( j \)th column of \( A \) with the column of constants on the right side of the equations.

Example Use Cramer’s rule to solve the following systems.

(a) \[
\begin{align*}
x_1 + 2x_2 + x_3 &= 1 \\
2x_1 + x_3 &= 2 \\
-x_1 + x_2 + 2x_3 &= 4
\end{align*}
\]

(b) \[
\begin{align*}
x_1 + 2x_2 + 3x_3 &= -1 \\
2x_1 + x_2 - 4x_3 &= 9 \\
x_1 - x_2 + 2x_3 &= -2
\end{align*}
\]

(c) \[
\begin{align*}
x_1 + x_3 - x_4 &= -4 \\
2x_1 + x_2 - x_3 + x_4 &= 8 \\
-x_1 + 2x_2 - 2x_4 &= -5 \\
x_1 + 2x_3 + 2x_4 &= 3
\end{align*}
\]

Assignment Do Problems 6.6: 2, 4, 6, 12, 18, 24, 32, 34, 36.