Unit 1
Limits

1. The concept of limits involves the notion of getting closer and closer to something, but yet not touching it.

2. We will let a variable “inch up” to a particular value and examine the effect it has on the values of a function.

3. **Definition 1** The limit of \( f(x) \) as \( x \) approaches \( a \) is the number \( L \), written

\[
\lim_{x \to a} f(x) = L
\]

provided that \( f(x) \) is arbitrarily close to \( L \) for all \( x \) sufficiently close to, but not equal to, \( a \). If there is no such number, we say that the limit does not exist.

4. Properties of Limits

(a) If \( f(x) = c \) is a constant function, then

\[
\lim_{x \to a} f(x) = \lim_{x \to a} c = c.
\]

(b) \( \lim_{x \to a} x^n = a^n \), for any positive integer \( n \)

If \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) exist, then

(c) \( \lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) \).

That is, the limit of a sum or difference is the sum or difference, respectively, of the limits.

(d) \( \lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) \).

That is, the limit of a product is the product of the limits.

(e) \( \lim_{x \to a} (cf(x)) = c \cdot \lim_{x \to a} f(x) \), where \( c \) is a constant.

That is, the limit of a constant times a function is the constant times the limit of the function.

If \( f \) is a polynomial function, then

\[
\lim_{x \to a} f(x) = f(a).
\]

(f) \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \) if \( \lim_{x \to a} g(x) \neq 0 \).

That is, the limit of a quotient is the quotient of limits, provided that the denominator does not have a limit of 0.

(g) \( \lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} \).

For this property, if \( n \) is even, we require that \( \lim_{x \to a} f(x) \) be positive.
(h) if \( f \) and \( g \) are two functions for which \( f(x) = g(x) \), for all \( x \neq a \), then

\[
\lim_{x \to a} f(x) = \lim_{x \to a} g(x).
\]

**Problem 1** Find the limits.

(a) \( \lim_{x \to 1} \frac{x^2 - 1}{x + 1} \)  \hspace{1cm} (b) \( \lim_{x \to 1} \frac{x^3 - 1}{x - 1} \)  \hspace{1cm} (c) \( \lim_{x \to 2} \frac{x^2 + 2x}{x + 2} \)

(d) \( \lim_{x \to -1} \frac{x + 1}{x + 1} \)  \hspace{1cm} (e) \( \lim_{x \to -3} \frac{x^4 - 81}{x^2 + 8x + 15} \)  \hspace{1cm} (f) \( \lim_{x \to 2} \frac{3x^2 - x - 10}{x^2 + 5x - 14} \)

**Problem 2** Find

\[
\lim_{x \to 6} \frac{\sqrt{x^2 - 2} - 2}{x - 6}.
\]

**Problem 3** Find the constant \( c \) so that

\[
\lim_{x \to 3} \frac{x^2 + x + c}{x^2 - 5x + 6}
\]

exists. For this value of \( c \), determine the limit.

5. A special limit:

\[
\lim_{x \to 0} (1 + x)^{1/x} = e \approx 2.718.
\]
Unit 2
Limits (continued)

1. Terms to consider: one-sided limits, infinite limits, limits at infinity.

2. The limit exists if and only if both one-sided limits exist and are equal.

**Problem 4 — One-Sided Limits and Infinite Limits.** Find the limit (if it exists).

(a) \( \lim_{x \to -1^+} \frac{2}{x + 1} \)

(b) \( \lim_{x \to 2^-} \frac{x^2 + 2x}{x^2 - 4} \)

**Problem 5 — Limits at Infinity.** Find the limit (if it exists).

(a) \( \lim_{x \to \infty} \frac{4}{(x - 5)^3} \)

(b) \( \lim_{x \to -\infty} \sqrt{4 - x} \)

3. We note the following:

\[ \lim_{x \to \pm \infty} \frac{1}{x^p} = 0 \quad \text{where } p > 0. \]

4. **Theorem 1 — Limits at Infinity for Rational Functions.** If \( f(x) \) is a rational function and \( a_nx^n \) and \( b_mx^m \) are the terms in the numerator and denominator, respectively, with the greatest powers of \( x \), then

\[ \lim_{x \to \pm \infty} f(x) = \lim_{x \to \pm \infty} \frac{a_nx^n}{b_mx^m}. \]

**Problem 6** Find the limit (if it exists).

(a) \( \lim_{x \to \infty} \frac{x^2 - 1}{7 - 2x + 8x^2} \)

(b) \( \lim_{x \to -\infty} \frac{x}{(3x - 1)^2} \)

(c) \( \lim_{x \to \infty} \frac{x^5 - x^4}{x^4 - x^3 + 2} \)

5. As \( x \to \infty \) (or \( x \to -\infty \)), the limit of a polynomial function is the same as the limit of its term that involves the greatest power of \( x \). For example, \( \lim_{x \to -\infty} (x^3 - x^2 + x - 2) = \lim_{x \to -\infty} x^3 = -\infty. \)
6. Limits for a Case-Defined Function

**Problem 7** If \( f(x) = \begin{cases} 
  x^2 + 1 & \text{if } x \geq 1 \\
  3 & \text{if } x < 1 
\end{cases} \), find the limit (if it exists):

\[
\lim_{x \to 1^+} f(x), \quad \lim_{x \to 1^-} f(x), \quad \lim_{x \to 1} f(x), \quad \lim_{x \to \infty} f(x), \quad \lim_{x \to -\infty} f(x).
\]

**Exercise 2** Problems 10.2: 2, 4, 8, 16, 18, 44, 52, 58, 64
Unit 3  
Continuity

1. **Definition 2** A function $f$ is **continuous** at $a$ if and only if the following three conditions are met:

   (a) $f(a)$ exists.
   
   (b) $\lim_{x \to a} f(x)$ exists.
   
   (c) $\lim_{x \to a} f(x) = f(a)$.

2. A polynomial function is continuous at every point.

3. **Discontinuities of a Rational Function**

   **Theorem 2** A rational function is discontinuous at points where the denominator is 0 and is continuous otherwise. Thus, a rational function is continuous on its domain.

**Problem 8** Determine whether the function $f(x) = \frac{x - 4}{x^2 - 16}$ is continuous at 4 and at -4.

**Problem 9** Determine whether the function $f(x) = \begin{cases} x + 2 & \text{if } x \geq 2 \\ x^2 & \text{if } x < 2 \end{cases}$ is continuous at 2 and at 0.

**Problem 10** Find all points of discontinuity of each function:

1. $f(x) = \frac{x^2 + 3x - 4}{x^2 - 4}$
2. $f(x) = \frac{x^4}{x^4 - 1}$
3. $f(x) = \begin{cases} \frac{16}{x^2} & \text{if } x \geq 2 \\ 3x - 2 & \text{if } x < 2 \end{cases}$

**Exercise 3** Problems 10.3: 2, 9, 12, 20, 30
1. A **secant line** is a line that intersects a curve at two or more points.

2. The **tangent line** to the curve at $P$ is defined to be the common limiting position of the secant lines joining the point $P$ with any other points of the curve.

3. The **slope of a curve** at a point $P$ is the slope, if it exists, of the tangent line at $P$.

4. **Definition 3** The slope of the tangent line at $(a, f(a))$ is given by

$$m_{\text{tan}} = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$ 

**Problem 11** Find the slope of the tangent line to the curve $y = f(x) = x^2$ at the point $(1, 1)$.

5. **Definition 4** The **derivative** of a function $f$ is the function denoted $f'$ and defined by

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

provided that this limit exists. If $f'(a)$ can be found, then $f$ is said to be **differentiable** at $a$, and $f'(a)$ is called the derivative of $f$ at $a$ or the derivative of $f$ with respect to $x$ at $a$. The process of finding the derivative is called **differentiation**.

6. Because the derivative gives the slope of the tangent line, $f'(a)$ is the slope of the line tangent to the graph of $y = f(x)$ at $(a, f(a))$.

7. If $f$ is differentiable at $a$, then $f$ is continuous at $a$. That is, differentiability at a point implies continuity at that point.

8. It is false that continuity implies differentiability. For example, consider the function $f(x) = |x|$. This function is continuous at 0 but not differentiable there.

**Problem 12** Use the definition of the derivative to find each of the following: (1) $f'(x)$ if $f(x) = 4x - 1$ (2) $\frac{dp}{dq}$ if $p = 3q^2 + 2q + 1$ (3) $\frac{d}{dx} \sqrt{x + 2}$

**Problem 13** Find an equation of the tangent line to the curve $y = \frac{3}{x - 1}$ at the point $(2, 3)$. 
Problem 14  Find an equation of the tangent line to the curve $y = (x - 7)^2$ at the point $(6, 1)$.

Exercise 4  Problems 11.1: 4, 8, 10, 12, 16, 18, 22, 28
Unit 5
Rules for Differentiation

Rule 1 Derivative of a Constant: If \( c \) is a constant, then

\[
\frac{d}{dx}(c) = 0.
\]

That is, the derivative of a constant function is zero.

Rule 2 Derivative of \( x^n \): If \( n \) is any real number, then

\[
\frac{d}{dx}(x^n) = nx^{n-1}.
\]

That is, the derivative of a constant power of \( x \) is the exponent times \( x \) raised to a power one less than the given power.

Rule 3 Constant Factor Rule: If \( f \) is a differentiable function and \( c \) is a constant, then \( cf(x) \) is differentiable, and

\[
\frac{d}{dx}(cf(x)) = cf'(x).
\]

That is, the derivative of a constant times a function is the constant times the derivative of the function.

Rule 4 Sum or Difference Rule: If \( f \) and \( g \) are differentiable functions, then \( f + g \) and \( f - g \) are differentiable, and

\[
\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x).
\]

That is, the derivative of the sum (difference) of two functions is the sum (difference) of their derivatives.

Problem 15 Differentiate the functions.

(a) \( f(x) = \sqrt{x} (\sqrt{x} - 6x + 3) \)
(b) \( f(x) = \frac{x}{7} + \frac{7}{x} \)
(c) \( f(x) = x^2(x - 2)(x + 4) \)
(d) \( f(x) = \frac{\sqrt{8x^2} + x}{6\sqrt{x}} \)

Problem 16 Find an equation of the tangent line to the curve.

(a) \( y = 3 + x - 5x^2 + x^4 \) when \( x = 0 \)
(b) \( y = \frac{\sqrt{x}(2 - x^2)}{x} \) when \( x = 4 \)
Problem 17  Find all points on the curve $y = \frac{x^5}{5} - x + 1$ where the tangent line is horizontal.

Exercise 5  Problems 11.2: even-numbered problems 2-74; 82, 85, 88
Unit 6

The Derivative as a Rate of Change

1. If \( y = f(x) \), then average rate of change of \( y \) with respect to \( x \) over the interval from \( x \) to \( x + \Delta x \) is

\[
\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x},
\]

and instantaneous rate of change of \( y \) with respect to \( x \) is

\[
\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.
\]

2. Applications of Rate of Change to Economics

(a) A manufacturer’s total-cost function, \( c = f(q) \), gives the total cost \( c \) of producing and marketing \( q \) units of a product.

(b) The rate of change of \( c \) with respect to \( q \) is called the marginal cost. Thus,

\[
\text{marginal cost} = \frac{dc}{dq}.
\]

(c) We interpret the marginal cost as the approximate cost of one additional unit of output.

(d) If \( c \) is a total cost of producing \( q \) units of a product, then the average cost per unit, \( \bar{c} \), is \( \bar{c} = \frac{c}{q} \).

(e) A manufacturer’s total-revenue function, \( r = f(q) \), gives the total monetary value received for selling \( q \) units.

(f) If \( p \) denotes the demand function giving the price per unit for selling \( q \) units, then the total revenue \( r \) is given by \( r = pq \).

(g) The rate of change of \( r \) with respect to \( q \) is called the marginal revenue. Thus,

\[
\text{marginal revenue} = \frac{dr}{dq}.
\]

(h) We interpret the marginal revenue as the approximate revenue received from selling one additional unit of output.

3. The relative rate of change of \( f(x) \) is \( \frac{f'(x)}{f(x)} \).

4. The percentage rate of change of \( f(x) \) is \( \frac{f'(x)}{f(x)} \cdot 100\% \).

Problem 18 For the cost function \( c = 0.3q^2 + 3.5q + 9 \), how fast does \( c \) change with respect to \( q \) when \( q = 10 \)? Determine the percentage rate of change of \( c \) with respect to \( q \) when \( q = 10 \).
Problem 19  For a certain manufacturer, the revenue obtained from the sale of $q$ units of a product is given by $r = 10q - 0.1q^2$.

(a) How fast does $r$ change with respect to $q$?
(b) When $q = 25$, find the relative rate of change of $r$ and to the nearest percent, find the percentage rate of change of $r$.

Problem 20  A manufacturer of mountain bikes has found that when 20 bikes are produced per day, the average cost is $150 and the marginal cost is $125. Based on that information, approximate the total cost of producing 21 bikes per day.

Exercise 6  Problems 11.3: 6, 9, 16, 22, 26, 40, 42
Unit 7
More Differentiation Rules

1. The Product Rule
   If $f$ and $g$ are differentiable functions, then the product $fg$ is differentiable, and
   \[ \frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x). \]
   That is,
   \[ \frac{d}{dx}(Hi \cdot Ho) = Hi \, di \, Ho + Ho \, di \, Hi. \]

2. The Quotient Rule
   If $f$ and $g$ are differentiable functions and $g(x) \neq 0$, then the quotient $f/g$ is also differentiable, and
   \[ \frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}. \]
   That is,
   \[ \frac{d}{dx} \left( \frac{Hi}{Ho} \right) = \frac{Ho \, di \, Hi - Hi \, di \, Ho}{Ho \, Ho}. \]

3. The Chain Rule
   If $y$ is a differentiable function of $u$ and $u$ is a differentiable function of $x$, then $y$ is a differentiable function of $x$ and
   \[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}. \]

4. The Power Rule
   If $u$ is a differentiable function of $x$ and $n$ is any real number, then
   \[ \frac{d}{dx} u^n = nu^{n-1} \frac{du}{dx}. \]

**Problem 21** Differentiate the functions.

(a) $y = (\sqrt{x} + 5x - 2) \left( \frac{\sqrt{x} - 3}{x} \right)$
(b) $y = \frac{8x^2 - 2x + 1}{2x^2 - 3x + 2}$
(c) $y = 3 - 12x^3 + \frac{1 - \frac{x^2 + 2}{5}}{x^2 + 5}$
(d) $y = (2x^3 - 8x)^{-12}$
(e) $y = \frac{(3x + 2)^5}{(4x - 5)^2}$
(f) $y = 3 \sqrt{(x - 2)^2(x + 2)}$
Problem 22  The cost $c$ of producing $q$ units of a product is given by

$$c = 5500 + 12q + 0.2q^2.$$  

If the price per unit $p$ is given by the equation

$$q = 900 - 1.5p,$$

then find the rate of change of cost with respect to price per unit when $p = 85$.

Problem 23  Suppose $y = f(x)$, where $x = g(t)$. Given that $g(2) = 3$, $g'(2) = 4$, $f(2) = 5$, $f'(2) = 6$, $g(3) = 7$, $g'(3) = 8$, $f(3) = 9$, and $f'(3) = 10$, determine the value of

$$\frac{dy}{dt} \bigg|_{t=2}.$$  

Exercise 7  Problems 11.4: even-numbered problems 24-46, 54, 56  
Problems 11.5: 2, 4, 6, 8, 10, 22, 32, 40, 52, 68
Unit 8
Derivatives of Logarithmic and Exponential Functions

1. Derivatives of Logarithmic Functions
   (a) \( \frac{d}{dx} \ln |x| = \frac{1}{x} \) for \( x \neq 0 \).
   (b) \( \frac{d}{dx} \ln |u| = \frac{1}{u} \cdot \frac{du}{dx} \) for \( u \neq 0 \).
   (c) \( \frac{d}{dx} \log_b |u| = \frac{1}{(\ln b)u} \cdot \frac{du}{dx} \) for \( u \neq 0 \).

Problem 24  Differentiate the functions.
   (a) \( y = \ln \left( \frac{2x + 3}{3x - 4} \right) \)  (b) \( y = \ln^2(2x + 11) \)
   (c) \( y = \ln(x^3 \sqrt{2x + 1}) \)  (d) \( y = \ln(x + \sqrt{1 + x^2}) \)

Problem 25  Find an equation of the tangent line to the curve \( y = x(\ln x - 1) \)
at the point where \( x = e \).

Problem 26  A manufacturer’s average-cost function, in dollars, is given by
   \[ \bar{c} = \frac{500}{\ln(q + 20)}. \]
   Find the marginal cost (rounded to two decimal places) when \( q = 50 \).

2. Derivatives of Exponential Functions
   (a) \( \frac{d}{dx} e^x = e^x \).
   (b) \( \frac{d}{dx} e^u = e^u \frac{du}{dx} \).
   (c) \( \frac{d}{dx} b^u = b^u (\ln b) \frac{du}{dx} \).

Problem 27  Differentiate the functions.
   (a) \( y = e^{2x}(x + 6) \)  (b) \( y = 2^{1+\sqrt{x}} \)
   (c) \( y = \frac{e^x - 1}{e^x + 1} \)  (d) \( y = e^{x^2 \ln x^2} \)
Problem 28  If \( f(x) = 5x^2 \ln x \), find \( f'(1) \).

Problem 29  If \( w = e^{x^3-4x} + x \ln(x - 1) \) and \( x = \frac{t + 1}{t - 1} \), find \( \frac{dw}{dt} \) when \( t = 3 \).

Problem 30  Calculate the relative rate of change of

\[ f(x) = 10^{-x} + \ln(8 + x) + 0.01e^{x^2} \]

when \( x = 2 \). Round your answer to four decimal places.

Exercise 8  Problems 12.1: even-numbered problems 2-44, 46, 48
Problems 12.2: even-numbered problems 2-28, 38, 42
Unit 9

Implicit Differentiation, Higher-Order Derivatives

1. Implicit Differentiation Procedure
   For an equation that we assume defines \( y \) implicitly as a differentiable function of \( x \), the derivative \( \frac{dy}{dx} \) can be found as follows:
   
   (a) Differentiate both sides of the equation with respect to \( x \).
   
   (b) Collect all terms involving \( \frac{dy}{dx} \) on one side of the equation, and collect all other terms on the other side.
   
   (c) Solve for \( \frac{dy}{dx} \).

2. If we differentiate \( f'(x) \), the resulting function \( (f')'(x) \) is called the second derivative of \( f \) at \( x \). It is denoted

   \[
   f''(x) \quad \text{or} \quad \frac{d^2}{dx^2}(f(x)),
   \]

   which is read “\( f \) double prime of \( x \).”

Continuing in this way, we get higher-order derivatives.

Problem 31 Find \( \frac{dy}{dx} \) by implicit differentiation.

(a) \( x^2 + xy - 2y^2 = 0 \)  
(c) \( x = \sqrt{y} + \sqrt[3]{y} \)  
(c) \( (1 + e^{3xy})^2 = 3 + \ln(x - y^2) \)

Problem 32 Find an equation of the tangent line to the curve \( x^3 + xy + y^2 = -1 \) at the point \((-1, 1)\).

Problem 33 Find the rate of change of \( q \) with respect to \( p \) for

\[
p = \frac{20}{(q + 5)^2}.
\]

Problem 34 If \( f(x) = 6x^3 - 12x^2 + 6x - 2 \), find all higher-order derivatives.

Problem 35 If \( y = e^{x^2} \), find \( \frac{d^2y}{dx^2} \).
Problem 36  If \( y = \frac{16}{x + 4} \), find \( \frac{d^2y}{dx^2} \) and evaluate it when \( x = 4 \).

Problem 37  If \( f(x) = x \ln x \), find the rate of change of \( f''(x) \).

Problem 38  Show that the equation
\[
f''(x) + 4f'(x) + 4f(x) = 0
\]
is satisfied if \( f(x) = (3x - 5)e^{-2x} \).

Problem 39  If \( c = 0.2q^2 + 2q + 500 \) is a cost function, how fast is marginal cost function changing when \( q = 97.357 \)?
Unit 10

Partial Derivatives

1. Definition 5  If \( z = f(x, y) \), the partial derivative of \( f \) with respect to \( x \), denoted \( f_x \) or \( \frac{\partial f}{\partial x} \), is the function of two variables, given by

\[
f_x(x, y) = \frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}
\]

provided that the limit exists.

the partial derivative of \( f \) with respect to \( y \), denoted \( f_y \) or \( \frac{\partial f}{\partial y} \), is the function of two variables, given by

\[
f_y(x, y) = \frac{\partial f}{\partial y}(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}
\]

provided that the limit exists.

2. Procedure to Find \( f_x(x, y) \) and \( f_y(x, y) \)
   To find \( f_x \), treat \( y \) as constants, and differentiate \( f \) with respect to \( x \) in the usual way.
   To find \( f_y \), treat \( x \) as constants, and differentiate \( f \) with respect to \( y \) in the usual way.

Problem 40  If \( f(x, y) = xy^2 + x^2y \), find \( f_x(x, y) \) and \( f_y(x, y) \). Also, find \( f_x(3, 4) \) and \( f_y(3, 4) \).

Problem 41  (a) If \( z = 3x^2y^3 - 9x^2y + xy^2 + 4y \), find \( \frac{\partial z}{\partial x} \), \( \frac{\partial z}{\partial y} \), \( \frac{\partial z}{\partial x} \) \( (1, 0) \), and

\[
\frac{\partial z}{\partial y} \bigg|_{(1,0)}.
\]

(b) If \( w = x^2e^{2x+3y} \), find \( \partial w/\partial x \) and \( \partial w/\partial y \).

Problem 42  If \( p = g(r, s, t, u) = \frac{rst}{rt^2 + st^2} \), find \( \frac{\partial p}{\partial s} \), \( \frac{\partial p}{\partial t} \), and \( \frac{\partial p}{\partial t} \) \( (0,1,1) \).

3. Applications of Partial Derivatives
   If \( z = f(x, y) \), then \( \frac{\partial z}{\partial x} \) is the rate of change of \( z \) with respect to \( x \) when \( y \) is held fixed.
Similarly, \( \frac{\partial z}{\partial y} \) is the rate of change of \( z \) with respect to \( y \) when \( x \) is held fixed.

**Problem 43**  A company manufactures two types of skis, the Lightning and the Alpine models. Suppose the joint-cost function for producing \( x \) pairs of the Lightning model and \( y \) pairs of the Alpine model per week is

\[
c = 0.07x^2 + 75x + 85y + 6000,
\]

where \( c \) is expressed in dollars. Determine the marginal costs \( \partial c/\partial x \) and \( \partial c/\partial y \) when \( x = 100 \) and \( y = 50 \), and interpret the results.

4. If the function \( P = f(\ell, k) \) gives the output \( P \) when the producer uses \( \ell \) units of labor and \( k \) units of capital, then this function is called a **production function**. We define the **marginal productivity with respect to \( \ell \)** to be \( \partial p/\partial \ell \). Likewise, the **marginal productivity with respect to \( k \)** is \( \partial p/\partial k \).

**Problem 44**  A manufacturer of a popular toy has determined that the production function is \( P = p \ell k \), where \( \ell \) is the number of labor-hours per week and \( k \) is the capital (expressed in hundreds of dollars per week) required for a weekly production of \( P \) gross of the toy. (One gross is 144 units.) Determine the marginal productivity functions, and evaluate them when \( \ell = 400 \) and \( k = 16 \). Interpret the results.

**Exercise 10**  Problems 17.1: 5, 6, 10, 17, 28, 35  
Problems 17.2: 2, 3, 4, 5, 10, 12
Unit 11
Extrema

1. A function \( f \) is said to be **increasing** on an interval \( I \) when, for any two numbers \( x_1, x_2 \) in \( I \), if \( x_1 < x_2 \), then \( f(x_1) < f(x_2) \). A function \( f \) is **decreasing** on an interval \( I \) when, for any two numbers \( x_1, x_2 \) in \( I \), if \( x_1 < x_2 \), then \( f(x_1) > f(x_2) \).

2. Criteria for Increasing and Decreasing Function
   Let \( f \) be differentiable on the interval \((a, b)\). If \( f'(x) > 0 \) for all \( x \) in \((a, b)\), then \( f \) is increasing on \((a, b)\). If \( f'(x) < 0 \) for all \( x \) in \((a, b)\), then \( f \) is decreasing on \((a, b)\).

3. A function \( f \) has a **relative maximum** at \( a \) if there is an open interval containing \( a \) on which \( f(a) \geq f(x) \) for all \( x \) in the interval. The relative maximum value is \( f(a) \).
   A function \( f \) has a **relative minimum** at \( a \) if there is an open interval containing \( a \) on which \( f(a) \leq f(x) \) for all \( x \) in the interval. The relative minimum value is \( f(a) \).

4. A function \( f \) has an **absolute maximum** at \( a \) if \( f(a) \geq f(x) \) for all \( x \) in the domain of \( f \). The absolute maximum value is \( f(a) \).
   A function \( f \) has an **absolute minimum** at \( a \) if \( f(a) \leq f(x) \) for all \( x \) in the domain of \( f \). The absolute minimum value is \( f(a) \).

5. A Necessary Condition for Relative Extrema
   If \( f \) has a relative extremum at \( a \), then \( f'(a) = 0 \) or \( f'(a) \) does not exist.

6. For \( a \) in the domain of \( f \), if either \( f'(a) = 0 \) or \( f'(a) \) does not exist, then \( a \) is called a **critical value** for \( f \). If \( a \) is a critical value, then the point \((a, f(a))\) is called a **critical point** for \( f \).

7. Criteria for Relative Extrema
   Suppose \( f \) is continuous on an open interval \( I \) that contains the critical value \( a \) and \( f \) is differentiable on \( I \), except possibly at \( a \).
   
   (a) If \( f'(x) \) changes from positive to negative as \( x \) increases through \( a \), then \( f \) has a relative maximum at \( a \).
   (b) If \( f'(x) \) changes from negative to positive as \( x \) increases through \( a \), then \( f \) has a relative minimum at \( a \).

**Problem 45**  Determine where the function is increasing or decreasing, and determine where relative maxima and minima occur.

(a) \( y = x^2 + 4x + 3 \)
(b) \( y = x^3 - \frac{5}{2}x^2 - 2x + 6 \)
(c) \( y = 3x^5 - 5x^3 \)
(d) \( y = \sqrt{x}(x - 2) \)
(e) \( y = \frac{x^2}{2 - x} \)

(f) \( y = \sqrt[3]{x^3 - 9x} \)

8. Extreme-Value Theorem

**Theorem 3** If a function is continuous on a closed interval, then the function has both a maximum value and a minimum value on that interval.

9. Procedure to Find Absolute Extrema for a Function \( f \) That is Continuous on \([a, b]\).

   **Step 1.** Find the critical values of \( f \).

   **Step 2.** Evaluate \( f(x) \) at the endpoints \( a \) and \( b \) and at the critical values in \((a, b)\).

   **Step 3.** The maximum value of \( f \) is the greatest of the values found in step 2. The minimum value of \( f \) is the least of the values found in step 2.

**Problem 46** Find the absolute extrema of the given function on the given interval.

(a) \( f(x) = -2x^2 - 6x + 5, \quad [-3, 2] \)

(b) \( f(x) = \frac{1}{4}x^4 - \frac{3}{2}x^2, \quad [0, 1] \)

(c) \( f(x) = 3x^4 - x^6, \quad [-1, 2] \)

(d) \( f(x) = \frac{x}{x^2 + 1}, \quad [0, 2] \)

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**Exercise 11** Problems 13.1: 2, 10, 14, 18, 20, 36, 54, 58

Problems 13.2: 1, 3, 5, 7, 11
Unit 12

Concavity
Applied Maxima and Minima

1. Let \( f \) be differentiable on the interval \((a, b)\). Then \( f \) is said to be **concave up** [**concave down**] on \((a, b)\) if \( f' \) is increasing [decreasing] on \((a, b)\).

2. **Criteria for Concavity**

   Let \( f' \) be differentiable on the interval \((a, b)\). If \( f''(x) > 0 \) for all \( x \) in \((a, b)\), then \( f \) is concave up on \((a, b)\). If \( f''(x) < 0 \) for all \( x \) in \((a, b)\), then \( f \) is concave down on \((a, b)\).

3. A function \( f \) has an **inflection point** at \( a \) if and only if \( f \) is continuous at \( a \) and \( f \) changes concavity at \( a \).

4. **Second-Derivative Test for Relative Extrema**

   **Theorem 4** Suppose \( f'(a) = 0 \).
   
   If \( f''(a) < 0 \), then \( f \) has a relative maximum at \( a \).
   
   If \( f''(a) > 0 \), then \( f \) has a relative minimum at \( a \).

5. **Guide for Solving Applied Max-Min Problems**

   **Problem 47** Sketch the curve.

   (a) \( y = x^3 - 9x^2 + 24x - 19 \)
   (b) \( y = x^3 - 25x^2 \)
   (c) \( y = 3x^4 - 4x^3 + 1 \)
   (d) \( y = x^2(x - 1)^2 \)
Problem 48  Find two nonnegative numbers whose sum is 20 and for which the product of twice one number and the square of the other number will be a maximum.

Problem 49  The demand equation for a monopolist’s product is \( p = -5q + 30 \). At what price will revenue be maximized.

Problem 50  For a monopolist’s product, the demand function is \( p = \frac{40}{\sqrt{q}} \) and the average-cost function is \( \bar{c} = \frac{1}{3} + \frac{2000}{q} \). Find the profit-maximizing price and output. At this level, show that marginal revenue is equal to marginal cost.

Problem 51  A container manufacturer is designing a rectangular box, open at the top and with a square base, that is to have a volume of 32 ft\(^3\). If the box is to require the least amount of material, what must be its dimensions?

Exercise 12  Problems 13.3: 41–50
Problems 13.6: 1, 4, 11, 12, 13
Unit 13
Integration

1. An antiderivative of a function $f$ is a function $F$ such that
   $$F'(x) = f(x).$$

2. Any two antiderivatives of a function differ only by a constant.

3. The indefinite integral of any function $f$ with respect to $x$ is written $\int f(x)\,dx$ and denotes the most general antiderivative of $f$. Hence,
   $$\int f(x)\,dx = F(x) + C \quad \text{if and only if} \quad F'(x) = f(x).$$

4. Basic Integration Formulas
   (a) $\int k\,dx = kx + C$, $k$ is a constant
   (b) $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$, $n \neq -1$
   (c) $\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \int \frac{dx}{x} = \ln |x| + C$ for $x \neq 0$
   (d) $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln |ax+b| + C$ for $a \neq 0$
   (e) $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$ for $a \neq 0$
   (f) $\int kf(x)\,dx = k \int f(x)\,dx$, $k$ is a constant
   (g) $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

5. Techniques of Integration
   (a) Guessing a likely antiderivative
   (b) Substitution

Problem 52  Determine the indefinite integrals.

(a) $\int \left(\sqrt{x} - \frac{3}{\sqrt{x}}\right)\,dx$
(b) $\int \left(e^{2x} - x^3(\sqrt{x} + 1)\right)\,dx$
(c) $\int \frac{(x^3 + 1)^2}{x^2}\,dx$
(d) $\int \frac{2x^2}{3 - 4x^3}\,dx$
(e) $\int \sqrt{x} + 1 \, dx$

(f) $\int \frac{x^3}{\sqrt{x^2 + 1}} \, dx$

(g) $\int (1 - e^{-x})^2 \, dx$

**Problem 53** A manufacturer has determined that the marginal-cost function is

$$\frac{dc}{dq} = 0.003q^2 - 0.4q + 40$$

where $q$ is the number of units produced. If marginal cost is $27.50$ when $q = 50$ and fixed costs are $5000$, what is the average cost of producing 100 units?

**Problem 54** The marginal-cost function for a manufacturer’s product is given by

$$\frac{dc}{dq} = \frac{9}{10} \sqrt{q} \sqrt{0.04q^{3/2} + 4}$$

where $c$ is the total cost in dollars when $q$ units are produced. Fixed costs are $360$.

(a) Determine the marginal cost when 25 units are produced.

(b) Find the total cost of producing 25 units.

**Exercise 13** Problems 14.2: 19, 37, 46, 52

Problems 14.3: 3, 6, 15

Problems 14.4: 43, 51, 74, 75

Problems 14.5: 21, 25, 40, 42, 52
1. Fundamental Theorem of Integral Calculus

If $f$ is continuous on the interval $[a, b]$ and $F$ is any antiderivative of $f$ on $[a, b]$, then

$$
\int_a^b f(x) \, dx = F(b) - F(a).
$$

2. Properties of the Definite Integral

(a) If $f$ is continuous and $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x) \, dx$ can be interpreted as the area of the region bounded by the curve $y = f(x)$, the $x$-axis, and the lines $x = a$ and $x = b$.

(b) $\int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx$ where $k$ is a constant

(c) $\int_a^b (f(x) \pm g(x)) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$

(d) $\int_a^b f(x) \, dx = \int_a^b f(t) \, dt$

The variable of integration is a dummy variable in the sense that any other variable produces the same result—that is, the same number.

(e) If $f$ is continuous on an interval $I$ and $a, b,$ and $c$ are in $I$, then

$$
\int_a^c f(x) \, dx = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx.
$$

(f) We define:

i. $\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$ if $a > b$

ii. $\int_a^a f(x) \, dx = 0$

**Problem 55** Determine the definite integrals.

(a) $\int_1^8 (x^{1/3} - x^{-1/3}) \, dx$

(b) $\int_{-1/3}^{20/3} \sqrt{3x + 5} \, dx$

(c) $\int_0^1 \frac{2x^3 + x}{x^2 + x^4 + 1} \, dx$

(d) $\int_0^3 \frac{3x}{\sqrt{4 - x}} \, dx$

(e) $\int_3^{27} 3(\sqrt{3x} - 2x + 1) \, dx$
(f) $\int_{1}^{2} 5x\sqrt{5-x^2} \, dx$

**Problem 56** Find the area of the region bounded by the graphs of the given equation.

(a) $y = 2x - 1$, $y = x^2 - 4$ and the vertical lines $x = 1$ and $x = 2$
(b) $y = (x - 1)^2$, $y = x - 1$
(c) $y = 10 - x^2$, $y = 4$
(d) $y = x^2$, $y = \sqrt{x}$
(e) $y = x^3 - 1$, $y = x - 1$

**Problem 57** A manufacturer’s marginal-cost function is $\frac{dc}{dq} = 0.6q + 2$. If production is presently set at $q = 80$ units per week, how much more would it cost to increase production to 100 units per week?

**Problem 58** The demand function for a product is $p = 100 - 0.05q$ where $p$ is the price per unit (in dollars) for $q$ units. The supply function is $p = 10 + 0.1q$. Determine consumers’ surplus and producers’ surplus under market equilibrium.

**Problem 59** The demand equation for a product is $p = (q - 4)^2$ and the supply equation is $p = q^2 + q + 7$ where $p$ (in thousands of dollars) is the price per 100 units when $q$ hundred units are demanded or supplied. Determine consumers’ surplus under market equilibrium.

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**Exercise 14** Problems 14.7: 12, 30, 39
Problems 14.9: 49, 51, 55
Problems 14.10: 3, 4