Notes on ICNS 103

Chapter 13: 13.3 Concavity, 13.4 The Second-Derivative Test, 13.5 Asymptotes, 13.6 Applied Maxima and Minima

1. Let \( f \) be differentiable on the interval \((a, b)\). Then \( f \) is said to be concave up [concave down] on \((a, b)\) if \( f' \) is increasing [decreasing] on \((a, b)\).

2. Criteria for Concavity

   Let \( f' \) be differentiable on the interval \((a, b)\). If \( f''(x) > 0 \) for all \( x \) in \((a, b)\), then \( f \) is concave up on \((a, b)\). If \( f''(x) < 0 \) for all \( x \) in \((a, b)\), then \( f \) is concave down on \((a, b)\).

3. A function \( f \) has an inflection point at \( a \) if and only if \( f \) is continuous at \( a \) and \( f \) changes concavity at \( a \).

4. Second-Derivative Test for Relative Extrema

   Suppose \( f'(a) = 0 \).
   If \( f''(a) < 0 \), then \( f \) has a relative maximum at \( a \).
   If \( f''(a) > 0 \), then \( f \) has a relative minimum at \( a \).

5. The line \( x = a \) is a vertical asymptote for the graph of the function \( f \) if and only if at least one of the following is true:

   \[
   \lim_{x \to a^+} f(x) = \pm \infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm \infty.
   \]

6. Vertical-Asymptote Rule for Rational Functions

   Suppose that \( f(x) = \frac{P(x)}{Q(x)} \) where \( P \) and \( Q \) are polynomial functions and the quotient is in lowest terms. The line \( x = a \) is a vertical asymptote for the graph of \( f \) if and only if \( Q(a) = 0 \) and \( P(a) \neq 0 \).

7. Let \( f \) be a function. The line \( y = b \) is a horizontal asymptote for the graph of \( f \) if and only if at least one of the following is true:

   \[
   \lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b.
   \]

8. A polynomial function of degree greater than 1 has no asymptotes.
Example Sketch the curve.

\begin{align*}
(a) \quad y &= \frac{x}{x - 1} \\
(b) \quad y &= \frac{1}{x^2 - 1} \\
(c) \quad y &= \frac{x^2 - 1}{x^3} \\
(d) \quad y &= \frac{3x}{(x - 2)^2}
\end{align*}


Step 1. When appropriate, draw a diagram that reflects the information in the problem.

Step 2. Set up a function for the quantity that you want to maximize or minimize.

Step 3. Express the function in step 2 as a function of one variable only, and note the domain of this function.

Step 4. Find the critical values of the function. After testing each critical value, determine which one gives the absolute extreme value you are seeking. If the domain of the function includes endpoints, be sure to also examine function values at these endpoints.

Step 5. Based on the results of step 4, answer the question(s) posed in the problem.

Example Find two nonnegative numbers whose sum is 20 and for which the product of twice one number and the square of the other number will be a maximum.

Example The demand equation for a monopolist’s product is \( p = -5q + 30 \). At what price will revenue be maximized.

Example For a monopolist’s product, the demand function is \( p = \frac{40}{\sqrt{q}} \) and the average-cost function is \( c = \frac{1}{3} + \frac{2000}{q} \). Find the profit-maximizing price and output. At this level, show that marginal revenue is equal to marginal cost.

Example A container manufacturer is designing a rectangular box, open at the top and with a square base, that is to have a volume of 32 ft\(^3\). If the box is to require the least amount of material, what must be its dimensions?