Notes on ICNS 103

Chapter 14: 14.6 The Definite Integral, 14.7 The Fundamental
Theorem of Integral Calculus, 14.9 Area, 14.10 Area between
Curves: Chapter 17: 17.1 Functions of Several Variables, 17.2 Partial
Derivatives, 17.3 Applications of Partial Derivatives

1. Fundamental Theorem of Integral Calculus

If \( f \) is continuous on the interval \([a, b]\) and \( F \) is any antiderivative of \( f \) on \([a, b]\), then
\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a).
\]

2. Properties of the Definite Integral

(a) If \( f \) is continuous and \( f(x) \geq 0 \) on \([a, b]\), then \( \int_{a}^{b} f(x) \, dx \) can be interpreted as the area of the region bounded by the curve \( y = f(x) \), the \( x \)-axis, and the lines \( x = a \) and \( x = b \).

(b) \( \int_{a}^{b} kf(x) \, dx = k \int_{a}^{b} f(x) \, dx \) where \( k \) is a constant

(c) \( \int_{a}^{b} (f(x) \pm g(x)) \, dx = \int_{a}^{b} f(x) \, dx \pm \int_{a}^{b} g(x) \, dx \)

(d) \( \int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(t) \, dt \)

The variable of integration is a dummy variable in the sense that any other variable produces the same result—that is, the same number.

(e) If \( f \) is continuous on an interval \( I \) and \( a, b, \) and \( c \) are in \( I \), then
\[
\int_{a}^{c} f(x) \, dx = \int_{a}^{b} f(x) \, dx + \int_{b}^{c} f(x) \, dx.
\]

Example Determine the definite integrals.

(a) \( \int_{0}^{1} (x^{1/3} - x^{-1/3}) \, dx \)
(b) \( \int_{-1/3}^{20/3} \sqrt{3x^3 + 5} \, dx \)
(c) \( \int_{0}^{1} \frac{2x^3 + x}{x^2 + x^4 + 1} \, dx \)

(d) \( \int_{1}^{2} 4xe^{2x} \, dx \)
(e) \( \int_{1}^{2} \frac{3x}{\sqrt{4 - x}} \, dx \)
(f) \( \int_{e}^{3} \sqrt{x} \ln(x^5) \, dx \)

(g) \( \int_{3}^{27} 3(\sqrt{3x} - 2x + 1) \, dx \)
(h) \( \int_{0}^{1} 3^{2x} \, dx \)
(i) \( \int_{1}^{2} 5x\sqrt{5 - x^2} \, dx \)
Example Find the area of the region bounded by the graphs of the given equation.

(a) \( y = x^2 + 1, \ x \geq 0, \ x = 0, \ y = 3 \)
(b) \( y = 10 - x^2, \ y = 4 \)
(c) \( y = x^2, \ y = 2, \ y = 5 \)
(d) \( y = x^3 - 1, \ y = x - 1 \)

3. Partial Derivatives

Procedure to Find \( f_x(x, y) \) and \( f_y(x, y) \)

To find \( f_x \), treat \( y \) as a constant, and differentiate \( f \) with respect to \( x \) in the usual way.

To find \( f_y \), treat \( x \) as a constant, and differentiate \( f \) with respect to \( y \) in the usual way.

Example Let \( g(x, y, z) = \frac{3x^2y^2 + 2xy + x - y}{xy - yz + xz} \). Find \( g_y(1, 1, 5) \).

Example If \( z = xe^{x-y} + ye^{y-x} \), show that
\[
\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = e^{x-y} + e^{y-x}.
\]

4. Applications of Partial Derivatives

(a) \( \frac{\partial z}{\partial x} \) is the rate of change of \( z \) with respect to \( x \) when \( y \) is held fixed.

(b) \( \frac{\partial z}{\partial y} \) is the rate of change of \( z \) with respect to \( y \) when \( x \) is held fixed.

(c) If the function \( P = f(\ell, k) \) gives the output \( P \) when the producer uses \( \ell \) units of labor and \( k \) units of capital, then this function is called a production function. We define the marginal productivity with respect to \( \ell \) to be \( \partial P/\partial \ell \). Likewise, the marginal productivity with respect to \( k \) is \( \partial P/\partial k \).

Example Let the joint-cost function be given by \( c = x\sqrt{x+y} + 5000 \). Find the marginal cost with respect to \( x \) when \( x = 40 \) and \( y = 60 \).

Example Suppose a production function is given by
\[
P = \frac{k\ell}{2k + 3\ell}.
\]

(a) Determine the marginal productivity functions.

(b) Show that when \( k = \ell \), the marginal productivities sum to \( \frac{1}{5} \).