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Notes on ICNS 103

Chapter 14: 14.6 The Definite Integral, 14.7 The Fundamental Theorem of Integral Calculus, 14.9 Area, 14.10 Area between Curves; Chapter 17: 17.1 Functions of Several Variables, 17.2 Partial Derivatives, 17.3 Applications of Partial Derivatives

1. Fundamental Theorem of Integral Calculus

   If f is continuous on the interval $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

2. Properties of the Definite Integral

   (a) If f is continuous and $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx$ can be interpreted as the area of the region bounded by the curve $y = f(x)$, the x−axis, and the lines $x = a$ and $x = b$.

   (b) $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ where k is a constant

   (c) $\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

   (d) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

   The variable of integration is a dummy variable in the sense that any other variable produces the same result–that is, the same number.

   (e) If f is continuous on an interval I and $a, b,$ and c are in I, then

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Example Determine the definite integrals.

(a) $\int_0^1 (x^{1/3} - x^{-1/3})\,dx$ (b) $\int_{-1/3}^{20/3} \sqrt{3x+5}\,dx$ (c) $\int_0^1 \frac{2x^3+x}{x^2+x^4+1}\,dx$

(d) $\int_1^2 4xe^{2x}\,dx$ (e) $\int_1^2 \frac{3x}{\sqrt{4-x}}\,dx$ (f) $\int_e^3 \sqrt[3]{x}\ln(x^5)\,dx$

(g) $\int_3^{27} 3(\sqrt{3x} - 2x + 1)\,dx$ (h) $\int_0^1 3^{2x}\,dx$ (i) $\int_1^2 5x\sqrt{5-x^2}\,dx$

<u>Example</u> Find the area of the region bounded by the graphs of the given equation.

(a) $y = x^2 + 1, \quad x \geq 0, \quad x = 0, \quad y = 3$
(b) $y = 10 - x^2, \quad y = 4$
(c) $y = x^2, \quad y = 2, \quad y = 5$
(d) $y = x^3 - 1, \quad y = x - 1$

3. Partial Derivatives

Procedure to Find $f_x(x, y)$ and $f_y(x, y)$

To find f_x, treat y as a constant, and differentiate f with respect to x in the usual way.

To find f_y, treat x as a constant, and differentiate f with respect to y in the usual way.

<u>Example</u> Let $g(x, y, z) = \dfrac{3x^2y^2 + 2xy + x - y}{xy - yz + xz}$. Find $g_y(1, 1, 5)$.

<u>Example</u> If $z = xe^{x-y} + ye^{y-x}$, show that

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = e^{x-y} + e^{y-x}.$$

4. Applications of Partial Derivatives

(a) $\dfrac{\partial z}{\partial x}$ is the rate of change of z with respect to x when y is held fixed.

(b) $\dfrac{\partial z}{\partial y}$ is the rate of change of z with respect to y when x is held fixed.

(c) If the function $P = f(\ell, k)$ gives the output P when the producer uses ℓ units of labor and k units of capital, then this function is called a <u>production function</u>. We define the marginal productivity with respect to ℓ to be $\partial P / \partial \ell$. Likewise, the marginal productivity with respect to k is $\partial P / \partial k$.

<u>Example</u> Let the joint-cost function be given by $c = x\sqrt{x + y} + 5000$. Find the marginal cost with respect to x when $x = 40$ and $y = 60$.

<u>Example</u> Suppose a production function is given by

$$P = \frac{k\ell}{2k + 3\ell}.$$

(a) Determine the marginal productivity functions.
(b) Show that when $k = \ell$, the marginal productivities sum to $\frac{1}{5}$.

<u>Exercises</u> Do Problems 14.7, 14.9, 14.10, 17.1, 17.2, 17.3.