Solutions to Homework 2

1.

$g(x) = \sqrt{2-3x}\ ;\ x = 0$

(i) g is defined at $x = 0$; $g(0) = \sqrt{2}$.

(ii) $\lim_{x\to 0} g(x) = \lim_{x\to 0} \sqrt{2-3x} = \sqrt{2}$, which exists

(iii) $\lim_{x\to 0} g(x) = \sqrt{2} = g(0)$

Thus g is continuous at $x = 0$.

2.

$$f(x) = \begin{cases} x+2 & \text{if } x \geq 2 \\ x^2 & \text{if } x < 2 \end{cases}$$

f is defined at $x = 2$ and $x = 0$; $f(2) = 4$, $f(0) = 0$.

Because $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x+2) = 4$ and

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 = 4$, we have

$\lim_{x\to 2} f(x) = 4$. In addition,

$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 = 0$. Since

$\lim_{x\to 2} f(x) = 4 = f(2)$ and $\lim_{x\to 0} f(x) = 0 = f(0)$,

f is continuous at both 2 and 0.

Answer: Continuous at 2 and 0.

3.

The denominator of this rational function is zero only when $x = \pm 2$. Thus f is discontinuous only at $x = \pm 2$.

4.

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

If $x < 1$, then $f(x) = \frac{1}{x}$, which is a rational function whose denominator is zero when $x = 0$. Thus f is discontinuous at $x = 0$. If $x > 1$, then $f(x) = 1$, which a polynomial function and hence continuous. At $x = 1$, f is defined $[f(x) = 1]$.

Because $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{1}{x} = 1$ and $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 1 = 1$, then $\lim_{x \to 1} f(x) = 1$.

Since $\lim_{x \to 1} f(x) = 1 = f(1)$, f is continuous at $x = 1$.

f is discontinuous at $x = 0$.

5.

$x^2 - 4 < 0$. $f(x) = x^2 - 4 = (x+2)(x-2)$ has zeros ± 2. By considering the intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$, we find $f(x) < 0$ on $(-2, 2)$.

Answer: $(-2, 2)$

6.

$(x + 5)(x + 2)(x - 7) \leq 0$

$f(x) = (x + 5)(x + 2)(x - 7)$ has zeros -5, -2 and 7. By considering the intervals $(-\infty, -5)$, $(-5, -2)$, $(-2, 7)$ and $(7, \infty)$, we find $f(x) < 0$ on $(-\infty, -5)$ and $(-2, 7)$.

Answer: $(-\infty, -5]$, $[-2, 7]$

7.

$f(x) = x^2 + 4$

$$y' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left[(x+h)^2 + 4\right] - \left[x^2 + 4\right]}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x + 0 = 2x$$

The slope at $(-2, 8)$ is $y'(-2) = 2(-2) = -4$.

8.

$3x^2 - 4$

$$y' = \lim_{h\to 0} \frac{[3(x+h)^2 - 4] - [3x^2 - 4]}{h}$$

$$= \lim_{h\to 0} \frac{6xh + 3h^2}{h} = \lim_{h\to 0}(6x + 3h) = 6x$$

If $x = 1$, then $y' = 6(1) = 6$.

The tangent line at $(1, -1)$ is $y + 1 = 6(x - 1)$ or $y = 6x - 7$.

9.

$f(x) = x^2 - x - 3$.

$$\frac{u}{dx}\left(x^2 - x - 3\right)$$

$$= \lim \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\left[(x+h)^2 - (x+h) - 3\right] - \left[x^2 - x - 3\right]}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2 - h}{h} = \lim_{h\to 0}(2x + h - 1) = 2x - 1$$

10.

$$f(x) = \frac{6}{x}$$

$$y' = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{6}{x+h} - \frac{6}{x}}{h}$$

Multiplying the numerator and denominator by $x(x + h)$ gives

$$y' = \lim_{h\to 0} \frac{6x - 6(x+h)}{hx(x+h)} = \lim_{h\to 0} \frac{-6h}{hx(x+h)}$$

$$= \lim_{h\to 0}\left[-\frac{6}{x(x+h)}\right] = -\frac{6}{x(x+0)} = -\frac{6}{x^2}$$