Solutions to Homework 2

1. 

\[ g(x) = \sqrt{2 - 3x} : x = 0 \]

(i) \( g \) is defined at \( x = 0 \): \( g(0) = \sqrt{2} \).

(ii) \( \lim_{x \to 0} g(x) = \lim_{x \to 0} \sqrt{2 - 3x} = \sqrt{2} \), which exists.

(iii) \( \lim_{x \to 0} g(x) = \sqrt{2} = g(0) \)

Thus \( g \) is continuous at \( x = 0 \).

2.

\[ f(x) = \begin{cases} 
  x + 2 & \text{if } x \geq 2 \\
  x^2 & \text{if } x < 2 
\end{cases} \]

\( f \) is defined at \( x = 2 \) and \( x = 0 \): \( f(2) = 4, f(0) = 0 \).

Because \( \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x + 2) = 4 \) and \( \lim_{x \to 2^-} x^2 = 4 \), we have

\[ \lim_{x \to 2^+} f(x) = 4 \]. In addition,

\[ \lim_{x \to 2^-} f(x) = 4 \].

Since \( \lim_{x \to 0} x^2 = 0 \).

\( f \) is continuous at both \( 2 \) and \( 0 \).

Answer: Continuous at \( 2 \) and \( 0 \).

3.

The denominator of this rational function is zero only when \( x = \pm 2 \). Thus \( f \) is discontinuous only at \( x = \pm 2 \).

4. 
5.

\[ f(x) = \begin{cases} \frac{1}{x} & \text{if } x < 1 \\ 1 & \text{if } x \geq 1 \end{cases} \]

If \( x < 1 \), then \( f(x) = \frac{1}{x} \), which is a rational function whose denominator is zero when \( x = 0 \). Thus \( f \) is discontinuous at \( x = 0 \). If \( x > 1 \), then \( f(x) = 1 \), which a polynomial function and hence continuous. At \( x = 1 \), \( f \) is defined \( f(1) = 1 \).

Because \( \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} \frac{1}{x} = 1 \) and

\[ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 1 \]

Since \( \lim_{x \to 1} f(x) = f(1) \), \( f \) is continuous at \( x = 1 \).

\[ f \] is discontinuous at \( x = 0 \).

5.

\[ x^2 - 4 < 0 \cdot f(x) = x^2 - 4 = (x + 2)(x - 2) \] has zeros \( \pm 2 \). By considering the intervals \(( -\infty, -2)\), \(( -2, 2)\), and \((2, \infty)\), we find \( f(x) < 0 \) on \(( -2, 2)\).

Answer: \((-2,2)\)

6.

\((x + 5)(x + 2)(x - 7) \leq 0\)

\( f(x) = (x + 5)(x + 2)(x - 7) \) has zeros \(-5, -2\) and \(7\). By considering the intervals \(( -\infty, -5)\), \(( -5, -2)\), \(( -2, 7)\) and \((7, \infty)\), we find \( f(x) < 0 \) on \(( -\infty, -5)\) and \(( -2, 7)\).

Answer: \((-\infty, -5), [-2, 7]\)

7.

\[ f'(x) = x^2 + 4 \]

\[ y' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

\[ = \lim_{h \to 0} \frac{[(x+h)^2 + 4] - [x^2 + 4]}{h} \]

\[ = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x + 0 = 2x \]

The slope at \((-2, 3)\) is \( y'(-2) = 2(-2) = -4 \).
8.

\[ y' = \lim_{h \to 0} \frac{[3(x + h)^2 - 4] - [3x^2 - 4]}{h} \]
\[ = \lim_{h \to 0} \frac{6xh + 3h^2}{h} = \lim_{h \to 0} (6x + 3h) = 6x \]

If \( x = 1 \), then \( y' = 6(1) = 6 \).

The tangent line at \((1, -1)\) is \( y + 1 = 6(x - 1) \) or \( y = 6x - 7 \).

9.

\[ f(x) = x^2 - x - 3. \]

\[ \frac{d}{dx}(x^2 - x - 3) \]
\[ = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]
\[ = \lim_{h \to 0} \frac{[ (x + h)^2 - (x + h) - 3] - [x^2 - x - 3]}{h} \]
\[ = \lim_{h \to 0} \frac{2xh + h^2 - h}{h} = \lim_{h \to 0} (2x + h - 1) = 2x - 1 \]

10.

\[ f(x) = \frac{6}{x} \]

\[ y' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{6}{x+h} - \frac{6}{x}}{h} \]

Multiplying the numerator and denominator by \( x(x + h) \) gives
\[ y' = \lim_{h \to 0} \frac{6x - 6(x + h)}{hx(x + h)} = \lim_{h \to 0} \frac{-6h}{hx(x + h)} \]
\[ = \lim_{h \to 0} \left[ -\frac{6}{x(x + h)} \right] = -\frac{6}{x(x + 0)} = -\frac{6}{x^2} \]