General Information

Instructor:
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Lecture:
Section 2: Monday 13.00 - 16.00 Room E3302
Section 401: Friday 16.00 - 19.00 Room E3402
Section 450: Tuesday 9.00 - 12.00 Room 6301

Downloadable Class Notes:
Class notes for all lessons can be downloaded from my website above.

Required Textbook:

Recommended Textbooks:

References:

Grade Breakdown:
Midterm Exam 45%
Final Exam 55%
Homework and Quiz 5% (Extra)

Evaluation:
\[ A \geq 80\% \quad 75 \leq B^+ < 80 \]
\[ 60 \leq B < 75 \quad 55 \leq C^+ < 60 \]
\[ 45 \leq C < 55 \quad 40 \leq D^+ < 45 \]
\[ 30 \leq D < 40 \quad F < 30 \]

Office Hours:
You are welcome to come into my office at any time.

Class Structure:
Three hours are divided into three sections of one hour. In each hour, the class structure is as follows:
50 minutes: Study new materials (or discuss homework due that day)
10 minutes: Break

Rules:
1) Close notes and books during exams. Open own notes and books during quizzes.
2) During exams and quizzes, only non-programmable calculator is allowed (Casio fx3800 or compatible.)
3) Homework problems are from the required textbook [1]. Each homework is due at next class. Homework is graded based on completion not correctness. Homework is an individual effort. You may consult your friends or me, but copying is strictly prohibited.
4) I do not accept late homework.
5) All in-class quizzes are random.
6) Grade sheet, which includes your score, ranking, and estimated grade, is posted on my website weekly.
7) Be punctual. If you are late, wait until next break to get in.
8) I do not accept students who transfer from other sections.
Course Schedule:

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Course Content:

Contents and homework problems are based on the required textbook [1].

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| Final Exam (Lesson 8 - 14) |
# 1 Introduction to Vibrations

Any motion that repeats itself after an interval of time is called vibration or oscillation.

Examples of vibration are:
- we hear because our eardrums vibrate
- we see because light waves undergo vibration
- breathing is vibration of lungs
- walking involves oscillatory motion of legs and hands
- we speak due to the oscillation of tongue
- destruction of Tacoma Narrows bridge

![Figure 1: Destruction of Tacoma Narrows bridge](image1)

- explosion of space shuttle Challenger

![Figure 2: Explosion of Challenger](image2)

A vibratory system includes mass (to store kinetic energy), spring (to store potential energy), and damper (means by which energy is gradually lost.)

## 1.1 Degrees of Freedom

Degrees of Freedom is the minimum number of independent coordinates required to determine completely the positions of all parts of a system at any instant of time.

For example,
- a free particle undergoing general motion in space has three degrees of freedom (x,y,z)
- a rigid body in space has six degrees of freedom (three components for position and three components for orientation)
- a continuous elastic body has infinite number of degrees of freedom

The coordinates necessary to describe the motion of a system constitute a set of generalized coordinates.

- Example 1: [3] Determine number of degrees of freedom and generalized coordinates of the following systems.

  a)
b) Solution 3 DOF, $\theta_1, \theta_2, \theta_3$.

c) Solution infinite degrees of freedom

d) Solution 2 DOF, $x, \theta$.

e) Solution 6 DOF, $x, y, z, \theta_1, \theta_2, \theta_3$.

f) Solution 7 DOF, $x, y, z, \theta_1, \theta_2, \theta_3, \theta_4$. 

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1.2 Categories of Vibrations

- **Discrete versus Continuous Systems**
  Systems with a finite number of degrees of freedom are called discrete systems or lumped parameter systems. Systems with an infinite number of degrees of freedom are called continuous systems or distributed systems.

  Most of the time, continuous systems are approximated as discrete systems, and solutions are obtained in a simpler manner.

- **Free versus Forced Vibrations**
  Free vibration is vibration of a system, after an initial disturbance, is left to vibrate on its own without external force acting on the system. Forced vibration is vibration of a system that is subjected to an external force.

  The system under free vibration will vibrate at one or more of its natural frequencies. When the system is excited (forced), the system is forced to vibrate at the excitation frequency.

- **Undamped versus Damped Vibrations**
  If no energy is lost or dissipated in friction or other resistance during oscillation, the vibration is known as undamped vibration. If any energy is lost in this way, it is called damped vibration.

- **Linear versus Nonlinear Vibrations**
  A linear system obeys principle of superposition shown in Figure 3. If the system does not obey principle of superposition, the system is said to be nonlinear system and hence having nonlinear vibration.

- **Deterministic versus Random Vibrations**
  Vibration that results from excitation (force or motion) that can be predicted at any given time is called deterministic vibration. If the excitation at a given time cannot be predicted, the vibration is called nondeterministic or random vibrations.

  ![Figure 4: Deterministic and random excitations](image)

2 Harmonic Motion

Harmonic motion is motion of an oscillating system that vibrates at one fixed frequency.

Consider Figure 5, the harmonic motion can be represented by a vector $\overrightarrow{OP}$ of magnitude $A$ rotating at a constant angular velocity $\omega$. The projection of the tip of the vector $\overrightarrow{OP}$ on the vertical axis is given by

$$y = A \sin \omega t,$$

and the projection on the horizontal axis is given by

$$x = A \cos \omega t.$$

If we replace the x-y plane with a complex plane where x-axis represents real axis and y-axis represents imaginary axis, we can write the vector $\overrightarrow{OP}$ in complex format as

$$\mathbf{Z} = x + iy = A \cos \omega t + i A \sin \omega t,$$

where $i = \sqrt{-1}$. 

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2.1 Exponential Form

Using Euler's equation

\[ e^{i\theta} = \cos \theta + i \sin \theta, \]

we can write the vector \( \overrightarrow{OP} \) in exponential form as

\[ \vec{Z} = A \cos \omega t + i A \sin \omega t \]

\[ = Ae^{i\omega t}. \]

2.1.1. Algebra of the Exponential Form

Let \( \vec{Z}_1 = A_1 e^{i\omega_1 t} \) and \( \vec{Z}_2 = A_2 e^{i\omega_2 t} \) be two different vectors in exponential form, we have

\[ \vec{Z}_1 + \vec{Z}_2 = A_1 e^{i\omega_1 t} + A_2 e^{i\omega_2 t}, \]

\[ \vec{Z}_1 - \vec{Z}_2 = A_1 e^{i\omega_1 t} - A_2 e^{i\omega_2 t}, \]

\[ \vec{Z}_1 \vec{Z}_2 = A_1 A_2 e^{i(\omega_1 + \omega_2) t}, \]

\[ \frac{\vec{Z}_1}{\vec{Z}_2} = \left( \frac{A_1}{A_2} \right) e^{i(\omega_1 - \omega_2) t}, \]

\[ (\vec{Z}_1)^n = A_1^n e^{i\omega_1 nt}. \]
Example 2: [3] Find the sum of the two harmonic motions
\[ x_1(t) = 10 \cos \omega t \quad \text{and} \quad x_2(t) = 15 \cos (\omega t + 2). \]
Solution

Let the sum of the two harmonic motions be

$$x(t) = A \cos(\omega t + \alpha)$$

$$= (\cos \omega t)(A \cos \alpha) - (\sin \omega t)(A \sin \alpha).$$

Then,

$$x(t) = x_1(t) + x_2(t)$$

$$=10 \cos \omega t + 15 \cos (\omega t + 2)$$

$$=10 \cos \omega t + 15 \left( \cos \omega t \cos 2 - \sin \omega t \sin 2 \right)$$

$$= (\cos \omega t)(10 + 15 \cos 2) - (\sin \omega t)(15 \sin 2).$$

Comparing (2) with (3), we have

$$A \cos \alpha = 10 + 15 \cos 2,$$

and

$$A \sin \alpha = 15 \sin 2.$$

From (4) and (5), we have

$$A = \sqrt{(10 + 15 \cos 2)^2 + (15 \sin 2)^2}$$

$$= 14.148,$$

$$\alpha = \cos^{-1} \frac{10 + 15 \cos 2}{14.148}$$

$$= 1.3 \text{ rad}.$$

Therefore, we have the sum as

$$x(t) = 14.148 \cos (\omega t + 1.3).$$

2.1.2. Velocity and Acceleration of Harmonic Motion

Consider a rotating vector \( \vec{Z} = A e^{i\omega t} \), its derivatives with respect to time are

$$\frac{d\vec{Z}}{dt} = i \omega A e^{i\omega t} = i \omega \vec{Z},$$

$$\frac{d^2\vec{Z}}{dt^2} = -\omega^2 A e^{i\omega t} = -\omega^2 \vec{Z}.\quad (6)$$

The projection of \( \vec{Z} = A e^{i\omega t} \) on the horizontal axis is \( x = A \cos \omega t \) whose derivatives are given as

$$\frac{dx}{dt} = -\omega A \sin \omega t = \omega A \cos \left( \omega t + \frac{\pi}{2} \right),$$

$$\frac{d^2x}{dt^2} = -\omega^2 A \cos \omega t = \omega^2 A \cos \left( \omega t + \pi \right).\quad (7)$$

We can see from both (6) and (7) that the acceleration vector leads the velocity vector by 90 degrees, and the velocity vector leads the displacement vector by 90 degrees as shown in Figure 6.

![Figure 6: Displacement, velocity, and acceleration vectors as rotating vectors.](image-url)
3 Periodic Motion

Periodic motion is motion of an oscillating system that vibrates at several different frequencies simultaneously. Examples of periodic motions are

- the vibration of a violin string is composed of the fundamental frequency \( f \) and all of its harmonics, \( 2f, 3f \), and so on
- free vibration of a multidegree-of-freedom system, to which the vibrations at each natural frequency contribute.

3.1 Fourier Series

A French mathematician J. Fourier (1768-1830) showed that any periodic signal can be approximated by a series of sines and cosines that are harmonically related.

Any periodic signal with period \( \tau = 2L \) can be approximated by the Fourier series

\[
f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),
\]

(8)

\( a_0, a_n, \) and \( b_n \) are called Fourier coefficients with following equations

\[
a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx,
\]

\[
a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx,
\]

(9)

\[
b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx.
\]

Example 3: [5] Find the Fourier series of the following periodic function

\[
f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}
\]

with \( f(x + 2\pi) = f(x) \)

\[
f(x)
\]

\[
\begin{array}{c}
-x \\
0 \\
\pi \\
2\pi
\end{array}
\]

\[
k
\]

\[
-k
\]

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Solution

In (8), the period is $2L$. Since the signal has period $2\pi$, we have

$$2L = 2\pi,$$
$$L = \pi.$$  

From (9), we have

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx$$
$$= \frac{1}{2\pi} \left( \int_{-\pi}^{0} -k \, dx + \int_{0}^{\pi} k \, dx \right)$$
$$= 0,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos \frac{n\pi x}{\pi} \, dx$$
$$= 0,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin \frac{n\pi x}{\pi} \, dx$$
$$= \frac{1}{\pi} \left( \int_{-\pi}^{0} -k \sin nx \, dx + \int_{0}^{\pi} k \sin nx \, dx \right)$$
$$= \frac{1}{\pi} \left( k \cos nx \bigg|_{-\pi}^{0} - \frac{k}{n} \cos nx \bigg|_{0}^{\pi} \right)$$
$$= \frac{1}{\pi} \left( k - \frac{k}{n} \cos(-n\pi) - \frac{k}{n} \cos n\pi + \frac{k}{n} \right)$$
$$= \frac{2k}{\pi n} (-\cos n\pi + 1).$$

3.1.1. Complex Form

The Fourier series can be written in complex form using the Euler’s equation (1) as follows

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x / L},$$
$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x / L} \, dx.$$

3.1.2. Gibbs Phenomenon

Figure 7 describes Gibbs phenomenon. As the number of terms of the Fourier series increases, the approximation can be seen to improve everywhere except near the discontinuity (point P.)

![Figure 7: Gibbs phenomenon.](image-url)
3.1.3. Frequency Spectrum

Frequency spectrum or spectral diagram is a plot of amplitude of each term in the Fourier series (8) with its corresponding circular frequency $\omega$.

If we expand (8), using $\omega = 2\pi / \tau$, we have

$$f(x) = a_0 + a_1 \cos \omega x + b_1 \sin \omega x$$
$$+ a_2 \cos 2\omega x + b_2 \sin 2\omega x$$
$$+ a_3 \cos 3\omega x + b_3 \sin 3\omega x + \ldots$$

Figure 8 shows an example of the frequency spectrum (using $d_n$ and $\phi_n$ instead of $a_n$ and $b_n$ respectively.)

![Figure 8: Frequency spectrum of a typical periodic function of time.](image)

The following plots show two examples of the frequency spectrum.
4 Vibration Terminology

Certain terminologies used in vibration analysis are as follows.

**Peak Value** generally means the maximum value of a vibrating body.

**Amplitude** is the maximum displacement of a vibrating body from its equilibrium position.

**Average Value** or **Mean Value** indicates a steady or static value. It can be found by the time integral

\[
\bar{x} = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t) \, dt.
\]

If the signal \( x(t) \) is periodic, the formula above reduces to

\[
\bar{x} = \frac{1}{\tau} \int_0^\tau x(t) \, dt, \quad (10)
\]

where \( \tau \) is the period of the signal \( x(t) \).

**Mean Square Value** is the average of the squared values, integrated over some time interval \( T \)

\[
\bar{x^2} = \lim_{T \to \infty} \frac{1}{T} \int_0^T x^2(t) \, dt.
\]

If the signal \( x(t) \) is periodic, the formula above reduces to

\[
\bar{x^2} = \frac{1}{\tau} \int_0^\tau x^2(t) \, dt, \quad (11)
\]

where \( \tau \) is the period of the signal \( x(t) \).

Example 4: [4] Find the mean value and the mean square value of the following signals:

a) \( x(t) = A \sin t \)

b) a rectified sine wave

\[ x(t) = A \sin t \]

\[ t \]
Solution

a) Since the signal \( x(t) = A \sin t \) has period \( 2\pi \), using (10), we get

\[
\bar{x} = \frac{1}{2\pi} \int_{0}^{2\pi} A \sin t \, dt
\]

\[
= 0.
\]

Using (11), we have

\[
\bar{x}^2 = \frac{1}{2\pi} \int_{0}^{2\pi} x^2(t) \, dt
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} A^2 \sin^2 t \, dt
\]

\[
= \frac{A^2}{2\pi} \int_{0}^{2\pi} \frac{1 - \cos 2t}{2} \, dt
\]

\[
= \frac{A^2}{2\pi} \left( \frac{1}{2} \left| \int_{0}^{2\pi} - \frac{1}{4} \sin 2t \right|_0^{2\pi} \right)
\]

\[
= \frac{A^2}{2\pi} (\pi - 0)
\]

\[
= \frac{A^2}{2}.
\]

b) Since the rectified sine wave has period \( \pi \), using (10), we get

\[
\bar{x} = \frac{1}{\pi} \int_{0}^{\pi} A \sin t \, dt
\]

\[
= \frac{A}{\pi} \left( -\cos t \right|_0^\pi = \frac{2A}{\pi}.
\]
Using (11), we get

\[ x^2 = \frac{1}{\pi} \int_0^\pi x^2(t) \, dt \]

\[ = \frac{1}{\pi} \int_0^\pi A^2 \sin^2 t \, dt \]

\[ = \frac{A^2}{\pi} \left( \frac{1 - \cos 2t}{2} \right) \bigg|_0^\pi \]

\[ = A^2 \left( \frac{\pi}{2} - 0 \right) \]

\[ = \frac{A^2}{2}. \]

Root Mean Square Value is the square root of the mean square value.

Decibel is a unit of measurement that is frequently used in vibration measurements. It is defined in terms of a power ratio

\[ dB = 10 \log_{10} \left( \frac{P_1}{P_2} \right). \]

Since power is proportional to the square of the amplitude or voltage, we have

\[ dB = 10 \log_{10} \left( \frac{x_1}{x_2} \right)^2 = 20 \log_{10} \left( \frac{x_1}{x_2} \right). \]

Thus an amplifier with a voltage gain of 5 has a decibel gain of

\[ 20 \log_{10}(5) = +14. \]

Octave is any frequency span with the maximum value twice the minimum value. For example, each of the ranges 80-160 Hz, 10-20 Hz, 30-60 Hz can be called an octave.

Cycle describes the movement of a vibrating body from its undisturbed or equilibrium position to its extreme position in one direction, then to the equilibrium position, then to its extreme position in the other direction, and back to equilibrium position.

Period of Oscillation is the time taken to complete one cycle of motion

\[ \tau = \frac{2\pi}{\omega}, \]

when \( \omega \) is called the circular frequency.

Frequency of Oscillation is the number of cycles per unit time

\[ f = \frac{1}{\tau} = \frac{\omega}{2\pi}. \]

Phase Difference generally means the phase difference between two signals. Consider

\[ x_1 = A_1 \sin \omega t, \]

\[ x_2 = A_2 \sin (\omega t + \phi), \]

the phase difference is \( \phi \) that means the maximum of the second vector would occur \( \phi \) radians earlier than that of the first vector.

Natural Frequency is the frequency of any system left to vibrate on its own without external force. A vibratory system having \( n \) degrees of freedom has \( n \) distinct natural frequencies of vibration.

Beating Phenomenon describes motion resulting from adding two harmonic motions with frequencies close to each other. For example, if we add the two signals...
\[ x_1(t) = A \cos \omega t, \]
\[ x_2(t) = A \cos (\omega + \delta)t, \]

where \( \delta \) is a small value, using the trigonometry identity

\[ \cos A + \cos B = 2 \cos \left( \frac{A + B}{2} \right) \cos \left( \frac{A - B}{2} \right), \]

we have

\[ x(t) = x_1(t) + x_2(t) = 2A \cos \frac{\delta t}{2} \cos \left( \omega + \frac{\delta}{2} \right)t. \]

The plot of \( x(t) \) is given in Figure 9.

![Figure 9: Beating phenomenon.](image)

**Lesson 1 Homework Problems**

1.43-1.47, 1.51, 1.52, 1.58, 1.66, 1.72


**References**