1 Vibration Analysis Procedure

The analysis of a vibrating system usually involves four steps: mathematical modeling, derivation of the governing equations, solution of the equations, and interpretation of the results.

**Step 1: Mathematical Modeling**

Mathematical model represents all the important features of the system for the purpose of deriving the mathematical equations governing the system’s behavior. The mathematical model should include enough details to be able to describe the system in terms of equations without making it too complex.

The mathematical model may be linear or nonlinear, depending on the behavior of the system’s components. Linear models permit quick solutions and are simple to handle; however, nonlinear models sometimes reveal certain characteristics of the system that cannot be predicted using linear models.

First, we can use a very crude or elementary model to get a quick insight into the overall behavior of the system. Subsequently, the model is refined by including more components and details so that the behavior of the system can be observed more closely.

- **Example 1:** [1] Consider a forging hammer shown in Figure 1. The forging hammer consists of a frame, a falling weight known as the tup, an anvil, and a foundation block. The anvil is a massive steel block on which material is forged into desired shape by the repeated blows of the tup. The anvil is usually mounted on an elastic pad to reduce the transmission of vibration to the foundation block and the frame. Come up with two versions of mathematical model of the forging hammer.

![Figure 1: A forging hammer.](image)
Example 2: [1] Consider a motorcycle with a rider in Figure 2. Develop a sequence of four mathematical models of the system. Consider the elasticity of the tires, elasticity and damping of the struts (in the vertical direction), masses of the wheels, and elasticity, damping, and mass of the rider.

Solution
Step 2: Derivation of Governing Equations

Once the mathematical model is available, we use the principles of dynamics and derive the equations that describe the vibration of the system.

The equations of motion can be derived by drawing the free-body diagrams of all the masses involved.

The equations of motion of a vibrating system are in the form of a set of ordinary differential equations for a discrete system and partial differential equations for a continuous system.

The equations may be linear or nonlinear, depending on the behavior of the components of the system.

Several approaches are commonly used to derive the governing equations. They are

- Newton’s second law of motion
- Energy method
- Equivalent system method (Rayleigh’s method)

Step 3: Solution of the Governing Equations

The equations of motion must be solved to find the response of the vibrating system. We can use the following techniques for finding the solution:

- Standard methods of solving differential equations
- Laplace transform methods
- Numerical methods
- Matrix methods

In this course, we will focus on the first three methods; each method will be discussed in the following lessons. The last method, namely, matrix methods are mostly used in higher degree of freedom and can be studied once the students are familiar with linear algebra.

Step 4: Interpretation of the Results

The solution of the governing equations gives the displacements, velocities, and accelerations of the various masses of the system. These results must be interpreted with a clear view of the purpose of the analysis and the possible design implications of the results.

2 Vibration Model: Mechanical Elements

As discussed before, a vibratory system consists of three elements: mass, spring, and damper.

2.1 Spring Elements

A spring is generally assumed to have negligible mass and damping. A force developed in the spring is given by

\[ F = kx, \]

where \( F \) is the spring force, \( x \) is the displacement of one end with respect to the other, and \( k \) is the spring stiffness or spring constant.
The work done in deforming a spring is stored as strain or potential energy in the spring and is given by

\[ U = \frac{1}{2} kx^2. \]

Actual springs are linear only up to a certain deformation. Beyond a certain value of deformation, the stress exceeds the yield point of the material and the force-deformation relation becomes nonlinear as in Figure 3.

\[ y(x) = \begin{cases} \frac{Px^2 (3a - x)}{6EI}, & 0 \leq x \leq a \\ \frac{Pa^2 (3x - a)}{6EI}, & a \leq x \leq l \end{cases} \]  

where \( E \) is Young’s modulus, \( I \) is moment of inertia of the cross section of the beam, and \( P \) is external force. Find the spring constant of a cantilever beam with an end mass \( m \) shown in Figure 5.
Solution

From (1), we have that the static deflection of the beam at the free end is given by

$$\delta_{st} = \frac{mgd^3 (3l - l)}{6EI} = \frac{mgd^3}{3EI}.$$

Therefore the spring constant is

$$k = \frac{mg}{\delta_{st}} = \frac{3EI}{d^3}.$$

$k$ can be called an equivalent stiffness of the cantilever beam.

2.1.1. Combination of Springs

Several springs are used in combination. They can be connected in parallel or in series or both. These springs can be combined into a single equivalent spring as follows.

Springs in Parallel

Consider two springs connected in parallel, from the free-body diagram we have

![Springs in Parallel Diagram]

Figure 6: Springs in parallel.
\[ W = k_1 \delta_1 + k_2 \delta_1 = k_{eq} \delta_1. \]

Therefore, the equivalent spring is

\[ k_{eq} = k_1 + k_2. \]

In general, if we have \( n \) springs with spring constants \( k_1, k_2, \ldots, k_n \) in parallel, then the equivalent spring constant \( k_{eq} \) can be obtained as

\[ k_{eq} = k_1 + k_2 + \ldots + k_n. \]

**Springs in Series**

Next, we consider two springs connected in series in Figure 7.

![Springs in Series](image)

Figure 7: Springs in series.

The static deflection of the system \( \delta_{st} \) is given by

\[ \delta_{st} = \delta_1 + \delta_2. \]

Since both springs are subjected to the same force \( W \), we have

\[ W = k_1 \delta_1, \quad W = k_2 \delta_2, \quad W = k_{eq} \delta_{st}. \]

Therefore,

\[ \delta_1 = \frac{k_{eq} \delta_{st}}{k_1} \quad \text{and} \quad \delta_2 = \frac{k_{eq} \delta_{st}}{k_2} \]

Substitute the equations above into (2), we get

\[ \frac{k_{eq} \delta_{st}}{k_1} + \frac{k_{eq} \delta_{st}}{k_2} = \delta_{st}, \]

therefore,

\[ \frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}. \]

In general, if we have \( n \) springs with spring constants \( k_1, k_2, \ldots, k_n \) in series, then the equivalent spring constant \( k_{eq} \) can be obtained as

\[ \frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots + \frac{1}{k_n}. \]

\[ \text{Example 4: [1]} \]

Figure 8 shows the suspension system of a freight truck with a parallel-spring arrangement. Find the equivalent spring constant of the suspension if each of the three helical springs is made of steel with a shear modulus \( G = 80 \times 10^9 \text{ N/m}^2 \) and has five effective turns, mean coil diameter \( D = 20 \text{ cm} \), and wire diameter \( d = 2 \text{ cm} \).
The equivalent spring constant of a helical spring under axial load is given by

\[ k_{eq} = \frac{Gd^4}{8nD^3}, \]

where \( d \) = wire diameter, \( D \) = mean coil diameter, \( n \) = number of active turns.

\[ k = \frac{Gd^4}{8nD^3} = \frac{(80 \times 10^9)(0.02)^4}{8(5)(0.2)^3} = 40,000 \text{ N/m}. \]

Since there are three springs connected in parallel, we have

\[ k_{eq} = 3k = 120,000 \text{ N/m}. \]

Example 5: [1] Determine the torsional spring constant of the steel propeller shaft shown in Figure 9. A hollow shaft under torsion has equivalent spring constant

\[ k_{eq} = \frac{\pi G}{32l} \left( D^4 - d^4 \right), \]

where \( l \) = length, \( D \) = outer diameter, \( d \) = inner diameter.
Solution

There are two sections of torsional springs, section 1-2 and section 2-3. Since $\delta_{12} + \delta_{23} = \delta_y$, we treat the two sections as springs connected in series. The spring constant of each section is given by

$$k_{12} = \frac{\pi \left( 80 \times 10^9 \right)}{32(2)} \left( 0.3^4 - 0.2^4 \right)$$

$$= 25.53 \times 10^6 \text{ Nm/rad},$$

$$k_{23} = \frac{\pi \left( 80 \times 10^9 \right)}{32(2)} \left( 0.25^4 - 0.15^4 \right)$$

$$= 8.90 \times 10^6 \text{ Nm/rad}.$$

Since the springs are connected in series, we have

$$\frac{1}{k_{eq}} = \frac{1}{k_{12}} + \frac{1}{k_{23}}$$

$$= \frac{1}{25.53 \times 10^6} + \frac{1}{8.90 \times 10^6},$$

$$k_{eq} = 6.60 \times 10^6 \text{ Nm/rad}.$$

Example 6: [1] Consider a crane in Figure 10(a). The boom AB has a cross-sectional area of 2,500 mm$^2$. The cable is made of steel with a cross-sectional area of 100 mm$^2$. Neglecting the effect of the cable CDEB, find the equivalent spring constant of the system in the vertical direction.
Figure 10: Crane lifting a load.
Solution

The equivalent system is given in Figure 10(c). First, we need to use trigonometry to find the length $l_1$ and the angle $\theta$. Using law of cosine, we have

$$l_1^2 = 3^2 + 10^2 - 2(3)(10)\cos 135^\circ,$$

$$l_1 = 12.31 \text{ m}.$$

The angle $\theta$ can be found using the law of cosine and the knowledge of $l_1$

$$10^2 = l_1^2 + 3^2 - 2(l_1)(3)\cos \theta,$$

$$\theta = 35.07^\circ.$$

A vertical displacement $x$ of point B will cause the spring $k_2$ to deform by an amount of $x_2 = x \cos 45^\circ$ and the spring $k_1$ to deform by an amount of $x_1 = x \cos (90^\circ - \theta)$.

The potential energy stored in the spring of the equivalent system is

$$U_{eq} = \frac{1}{2} k_{eq} x^2.$$ 

The potential energy stored in the springs of the actual system is

$$U = \frac{1}{2} k_2 \left( x \cos 45^\circ \right)^2 + \frac{1}{2} k_1 \left[ x \cos (90^\circ - \theta) \right]^2,$$

where
\[ k_1 = \frac{A_1E_1}{l_1} = \frac{(100 \times 10^{-6})(207 \times 10^6)}{12.31} = 1.68 \times 10^6 \text{ N/m}, \]
\[ k_2 = \frac{A_2E_2}{l_2} = \frac{(2500 \times 10^{-6})(207 \times 10^6)}{10} = 5.17 \times 10^7 \text{ N/m}. \]

Since the total potential energy stored in the springs of the equivalent system and the actual system must be equal, we have
\[ U_{eq} = U, \]
\[ k_{eq} = 26.43 \times 10^6 \text{ N/m}. \]

### 2.2 Mass or Inertia Elements

In many practical applications, several masses appear in combination. For a simple analysis, we can replace these masses by a single equivalent mass.

If the system has translational masses, we can replace them with a single equivalent mass. If the system has rotational masses, we can replace them with a single equivalent inertia. And if the system has both translational and rotational masses, we can choose to replace them with a single equivalent mass or a single equivalent inertia.

We can do this using the fact that the kinetic energy of the actual system and its equivalent system must be equal.

**Example 7:** [1] Figure 11(a) shows a system whose equivalent system is given in Figure 11(b). Find the equivalent mass \( m_{eq} \).
Solution

Let \( T \) be the kinetic energy of the actual system and let \( T_{eq} \) be the kinetic energy of the equivalent system. We have

\[
T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2
\]

\[
= \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \left( \frac{l_2 \dot{x}_1}{l_1} \right)^2 + \frac{1}{2} m_3 \left( \frac{l_3 \dot{x}_1}{l_1} \right)^2,
\]

\[
T_{eq} = \frac{1}{2} m_{eq} \dot{x}_1^2.
\]

Therefore,

\[
T = T_{eq},
\]

\[
\frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \left( \frac{l_2 \dot{x}_1}{l_1} \right)^2 + \frac{1}{2} m_3 \left( \frac{l_3 \dot{x}_1}{l_1} \right)^2 = \frac{1}{2} m_{eq} \dot{x}_1^2,
\]

\[
m_{eq} = m_1 + \left( \frac{l_2}{l_1} \right)^2 m_2 + \left( \frac{l_3}{l_1} \right)^2 m_3.
\]

Example 8: [1] Consider translational and rotational masses coupled in Figure 12.

a) Find equivalent translational mass
b) Find equivalent rotational inertia
Solution
a) Let $T$ be the kinetic energy of the actual system and let $T_{eq}$ be the kinetic energy of the equivalent system. We have

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2$$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \left( \frac{\dot{x}}{R} \right)^2,$$

$T_{eq} = \frac{1}{2} m_{eq} \dot{x}^2.$

Therefore,

$$T = T_{eq},$$

$$\frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \left( \frac{\dot{x}}{R} \right)^2 = \frac{1}{2} m_{eq} \dot{x}^2,$$

$$m_{eq} = m + \frac{J_0}{R^2}.$$

b) Let $T$ be the kinetic energy of the actual system and let $T_{eq}$ be the kinetic energy of the equivalent system. We have

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2$$

$$= \frac{1}{2} m (R \dot{\theta})^2 + \frac{1}{2} J_0 \dot{\theta}^2,$$

$T_{eq} = \frac{1}{2} J_{eq} \dot{\theta}^2.$

Therefore,

$$T = T_{eq},$$

$$\frac{1}{2} m (R \dot{\theta})^2 + \frac{1}{2} J_0 \dot{\theta}^2 = \frac{1}{2} J_{eq} \dot{\theta}^2,$$

$$J_{eq} = mR^2 + J_0.$$

Example 9: [1] Find the equivalent mass of the system in Figure 13.
Solution

Let $T$ be the kinetic energy of the actual system and let $T_{eq}$ be the kinetic energy of the equivalent system. Neglecting the rotational kinetic energy of link 2, we have

$$T = \frac{1}{2} m x^2 + \frac{1}{2} J_p \dot{\theta}_p^2 + \left( \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} m_1 \dot{x}_1^2 \right)$$

$$+ \frac{1}{2} m_2 x_2^2 + \left( \frac{1}{2} J_c \dot{\theta}_c^2 + \frac{1}{2} m_c \dot{x}_2^2 \right)$$

$$= \frac{1}{2} m x^2 + \frac{1}{2} J_p \left( \frac{\dot{x}}{r_p} \right)^2 + \left[ \frac{1}{2} \frac{m l_1^2}{12} \left( \frac{\dot{x}}{r_p} \right)^2 + \frac{1}{2} \frac{m_1 \dot{l}_1}{r_p} \right]$$

$$+ \frac{1}{2} m_2 \left( \frac{\dot{x}_2}{r_p} \right)^2 + \left[ \frac{1}{2} \frac{m_c r^2}{2} \left( \frac{\dot{x}_2}{r_p r_c} \right)^2 + \frac{1}{2} \frac{m_c \dot{l}_1}{r_p} \right],$$

$$T_{eq} = \frac{1}{2} m_{eq} \dot{x}^2.$$

Therefore,

$$\frac{1}{2} m_{eq} \dot{x}^2 = \frac{1}{2} m x^2 + \frac{1}{2} J_p \left( \frac{\dot{x}}{r_p} \right)^2 + \left[ \frac{1}{2} \frac{m l_1^2}{12} \left( \frac{\dot{x}}{r_p} \right)^2 + \frac{1}{2} \frac{m_1 \dot{l}_1}{r_p} \right]$$

$$+ \frac{1}{2} m_2 \left( \frac{\dot{x}_2}{r_p} \right)^2 + \left[ \frac{1}{2} \frac{m_c r^2}{2} \left( \frac{\dot{x}_2}{r_p r_c} \right)^2 + \frac{1}{2} \frac{m_c \dot{l}_1}{r_p} \right],$$

$$m_{eq} = m + \frac{J_p}{r_p^2} + \frac{m l_1^2}{12 r_p^2} + \frac{m c r^2}{4 r_p^2} + \frac{m l_1^2}{r_p^2} + \frac{m l_1^2}{2 r_p^2} + \frac{m l_1^2}{r_p^2}.$$
Damping is a mechanism by which the vibrating energy is gradually converted into other energy such as heat or sound. There are three common types of damping: viscous damping, Coulomb or dry friction damping, and hysteretic damping.

Viscous damping dissipates energy by using the resistance offered by the fluid such as air, gas, water, or oil.

Coulomb or dry friction damping is caused by friction between dry rubbing surfaces. It has constant magnitude but opposite in direction to the motion of the vibrating body.

Hysteretic damping happens when some materials are deformed and the vibrating energy are absorbed and dissipated by the material.

Among the three types of damping, viscous damping is the most commonly used and can be modeled as Figure 14.

\[ F = c \nu, \quad (4) \]

where \( c \) is called damping constant. Comparing (3) with (4), we have

\[ c = \frac{\mu A}{h}. \quad (5) \]

Combination of dampers is done similar to that of springs. The kinetic energy due to damping is given by

\[ U = \frac{1}{2} cv^2. \]

Example 10: [1] A bearing, which can be approximated as two flat plates separated by a thin film of lubricant, offers a resistance of 400 N when the relative velocity between the plates is 10 m/s. The area of the plates is 0.1 m\(^2\). The absolute viscosity of the oil in-between is 0.3445 Pa\(\cdot\)s. Determine the clearance between the plates.

From Newton's law of viscous flow, we have

\[ \tau = \frac{F}{A} = \mu \frac{du}{dy} = \frac{\mu \nu}{h}, \quad (3) \]

where \( \mu \) is the absolute viscosity of the fluid.
Figure 15: Flat plates separated by thin film of lubricant.

\[ \text{Area (A)} \]

\[ v \]

\[ h \]

Solution

From (4),

\[ F = cv, \]

\[ 400 = c(10), \]

\[ c = 40 \text{ Ns/m}. \]

From (5),

\[ c = \frac{\mu A}{h}, \]

\[ 40 = \frac{0.3445 \times 0.1}{h}, \]

\[ h = 0.86 \text{ mm}. \]

Example 11: [1] A milling machine is supported on four shock mounts as shown in Figure 16(a). The elasticity and damping of each shock mount can be modeled as a spring and a viscous damper as shown in Figure 16(b). Find the equivalent spring constant \( k_{eq} \) and the equivalent damping constant \( c_{eq} \) of the milling machine support.
Figure 16: A milling machine.
From the free-body diagram of the actual system in Figure 16(c), we have

\[ F_s = F_{s1} + F_{s2} + F_{s3} + F_{s4} = k_1x + k_2x + k_3x + k_4x, \]

\[ F_d = F_{d1} + F_{d2} + F_{d3} + F_{d4} = c_1\ddot{x} + c_2\ddot{x} + c_3\ddot{x} + c_4\ddot{x}. \]

From the free-body diagram of the equivalent system in Figure 16(d), we have

\[ F_s = k_{eq}x, \]

\[ F_d = c_{eq}\ddot{x}. \]

Since the forces of the two systems are equal, we have

\[ k_{eq} = k_1 + k_2 + k_3 + k_4, \]

\[ c_{eq} = c_1 + c_2 + c_3 + c_4. \]
Let the natural frequency be $\omega_n$ and let

$$\omega_n^2 = \frac{k}{m}, \quad (7)$$

we have

$$\ddot{x} + \omega_n^2 x = 0.$$

The solution of the second-order differential equation above is

$$x = A \sin \omega_n t + B \cos \omega_n t.$$

The constants $A$ and $B$ are evaluated from initial conditions $x(0)$ and $\dot{x}(0)$ to be

$$x = \frac{\dot{x}(0)}{\omega_n} \sin \omega_n t + x(0) \cos \omega_n t.$$

One remark is that when $x$ is chosen to be from the static equilibrium position, the weight $mg$ is cancelled with $k\Delta$ and disappears from the differential equation.

### 4 Natural Frequency

- Example 12: [1] A 0.25 kg mass is suspended by a spring having a stiffness of 0.1533 N/mm. Determine its natural frequency in cycles per second. Determine its static deflection.

**Solution**

From (7), we have

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.1533}{0.25}} = 0.783 \text{ rad/s}.$$  

From (6), we have

$$\Delta = \frac{mg}{k} = \frac{0.25(9.81)}{0.1533} = 16 \text{ mm}.$$  

- Example 13: [2] Determine the natural frequency of the mass $m$ on the end of a cantilever beam of negligible mass shown in Figure 18.

![Figure 18: A cantilever beam with end mass.](image)
Solution

A cantilever beam with end load as an equivalent spring constant

\[ k_{eq} = \frac{3EI}{l^3}. \]

Therefore, the natural frequency is

\[ \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3EI}{ml^3}} \text{ rad/s}. \]

Example 14: [2] A rod in Figure 19 has 0.5 cm in diameter and 2 m long. When the disk is given an angular displacement and released, it makes 10 oscillations in 30.2 seconds. Determine the polar moment of inertia of the disk.

Solution

From the Newton's second law, we have

\[ J\ddot{\theta} = -k_{eq}\theta. \]

For a shaft under torsion, the equivalent spring constant is

\[ k_{eq} = \frac{\pi Gd^4}{32l} = \frac{\pi (80 \times 10^9) (0.5 \times 10^{-2})^4}{32(2)} = 2.455 \text{ Nm/rad}. \]

From (7), we have

\[ \omega_n^2 = \frac{k}{J}, \]

\[ \left(2\pi \frac{10}{30.2}\right)^2 = \frac{2.455}{J}, \]

\[ J = 0.567 \text{ kg m}^2. \]
Example 15: [2] Consider a pivoted bar in Figure 20. The bar is horizontal in the equilibrium position with spring forces $P_1$ and $P_2$. Determine its natural frequency.

![Figure 20: A pivoted bar.](image)

**Solution**

From Newton’s second law, we have

$$J_0 \ddot{\theta} = -(ka\theta) a - (kb\theta) b,$$

$$\ddot{\theta} + \frac{ka^2 + kb^2}{J_0} \theta = 0,$$

$$\omega_n = \sqrt{\frac{ka^2 + kb^2}{J_0}}.$$
Lesson 2 Homework Problems

1.3, 1.4, 1.6, 1.7, 1.8, 1.30, 1.32, 1.34, 1.35

Homework problems are from the required textbook (Mechanical Vibrations, by Singiresu S. Rao, Prentice Hall, 2004)

References