1 Equations of Motion 3: Equivalent System Method

In systems in which masses are joined by rigid links, levers, or gears and in some distributed systems, various springs, dampers, and masses can be expressed in terms of one coordinate $x$ at a specific point and the system is simply transformed into a single DOF system.

This method is called the equivalent system method used to simplify higher DOF problems. We already discussed this method for systems of rigid bodies in Lesson 2. In this lesson, we present two more examples of distributed systems.

Example 1: [1] Consider a spring-mass system in equilibrium below. Assuming that the spring has mass $m_s$ per unit length $z$, use the equivalent system method to find the equivalent mass of the system and determine its effect on the natural frequency.

Solution

For the equivalent system, the kinetic energy is

$$T_{eq} = \frac{1}{2} m_{eq} \dot{x}^2.$$

The kinetic energy of the actual system is given by

$$T = \frac{1}{2} \int_0^z \left( \frac{y}{z} \right)^2 \frac{m_s}{z} dy + \frac{1}{2} m \dot{x}^2 = \frac{1}{2} \left( \frac{m_s}{3} + m \right) \dot{x}^2.$$

Therefore, the equivalent mass is $m_{eq} = \frac{m_s}{3} + m$, and the natural frequency of the system is

$$\omega_n = \frac{k}{\sqrt{\frac{m_s}{3} + m}}.$$
Example 2: [1] A simply supported beam has total mass \( m_b \). Supposed that the equivalent spring stiffness of the beam is \( 48EI/l^3 \) and the deflection under the load due to a concentrated force applied at midspan is 
\[ y = y_{\text{max}} \left[ \frac{3x}{l} - 4 \left( \frac{x}{l} \right)^3 \right], \]
determine the effective mass of the system at midspan and find its fundamental frequency.

\[ \frac{3x}{l} - 4 \left( \frac{x}{l} \right)^3 \]

Solution

The effective system has kinetic energy
\[ T_{\text{eq}} = \frac{1}{2} m_{\text{eq}} \dot{y}_{\text{max}}^2, \]
where \( y_{\text{max}} \) is the maximum deflection at midspan.

The actual system has kinetic energy
\[
T = \frac{1}{2} \int_0^l m_b \left\{ \dot{y}_{\text{max}} \left[ \frac{3x}{l} - 4 \left( \frac{x}{l} \right)^3 \right] \right\}^2 dx + \frac{1}{2} M\dot{y}_{\text{max}}^2
\]
\[ = \frac{1}{2} (0.4857 m_b + M) \dot{y}_{\text{max}}^2. \]

Therefore, the equivalent mass is \( m_{\text{eq}} = 0.4857 m_b + M \), and the natural frequency is 
\[ \omega_n = \sqrt{\frac{48EI}{l^3 (0.4857 m_b + M)}}. \]
2 Equations of Motion 4: Virtual Work Method

The virtual work method is another scalar method besides the work and energy method. It is useful especially for systems of interconnected bodies of higher DOF.

The principle of virtual work states that If a system in equilibrium under the action of a set of forces is given a virtual displacement, the virtual work done by the forces will be zero.

Example 3: [1] Use the virtual work method, determine the equation of motion for the system below.

Solution

Draw the system in the displaced position \( x \) and place the forces acting on it, including inertia and gravity forces. Give the system a small virtual displacement \( \delta x \) and determine the work done by each force. Using the fact that virtual work done by external forces equals virtual work done by inertia forces, we then obtain the equation of motion for the system.

The virtual work done by inertia forces is

\[ \delta W = m \ddot{x} \delta x. \]

The virtual work done by external forces is

\[ \delta W = -kx \delta x. \]

Equating the two quantities above and canceling \( \delta x \), we have the equation of motion

\[ m \ddot{x} + kx = 0. \]
Example 4: [1]

Solution

Draw the beam in the displaced position $\theta$ and place the forces acting on it, including the inertia and damping forces. Give the beam a virtual displacement $\delta \theta$ and determine the work done.

The virtual work done by inertia forces is

$$\delta W = \left( \frac{Ml^2}{3} \dot{\theta} \right) \delta \theta.$$

The virtual work done by external forces is

$$\delta W = -(cl\dot{\theta})l \delta \theta - \left( kl \frac{l}{2} \theta \right) \frac{l}{2} \delta \theta + \int_0^l x(p_0f(t)dx) \delta \theta$$

$$= -(cl\dot{\theta})l \delta \theta - \left( kl \frac{l}{2} \theta \right) \frac{l}{2} \delta \theta + p_0f(t)\frac{l^2}{2} \delta \theta.$$

Therefore, we have the equation of motion

$$\left( \frac{Ml^2}{3} \right) \ddot{\theta} + (cl^2) \dot{\theta} + k \frac{l^2}{4} \theta = p_0 \frac{l^2}{2} f(t).$$
Example 5: [1] Two simple pendulums are connected together with the bottom mass restricted to vertical motion in a frictionless guide as shown below. Using the virtual work method, determine the equation of motion.
Solution

First, we find acceleration of mass $m_2$ by using the relative motion analysis with translating axes using the formula

$$\ddot{a}_B = \ddot{a}_A + \ddot{a}_{B/A}.$$  

Placing point $A$ on the translating axes, the formula above in the vertical direction becomes

$$\left(\ddot{a}_B\right)_x = \left(\ddot{a}_A\right)_x + \left(\ddot{a}_{B/A}\right)_x$$

$$l\ddot{\theta}^2 \cos \theta + l\dot{\theta} \sin \theta = a_2 - l\ddot{\theta}^2 \cos \theta - l\dot{\theta} \sin \theta$$

$$a_2 = 2(l\ddot{\theta}^2 \cos \theta + l\dot{\theta} \sin \theta).$$

The virtual work done by inertia forces become

$$\delta W = \left(m_1l\ddot{\theta} \right)(l\delta \theta) + m_2 \left(2(l\ddot{\theta}^2 \cos \theta + l\dot{\theta} \sin \theta)\right)(2l\delta \theta \sin \theta).$$

The virtual work done by external forces become

$$\delta W = -m_2g \left(2l\delta \theta \sin \theta\right) - m_1g \left(l\delta \theta \sin \theta\right).$$

Therefore, the equation of motion becomes

$$0 = m_1l^2 \ddot{\theta} + m_2 \left(2(l\ddot{\theta}^2 \cos \theta + l\dot{\theta} \sin \theta)\right)(2l \sin \theta)$$

$$+ m_2g \left(2l \sin \theta\right) + m_1g \left(l \sin \theta\right).$$

If $\theta$ is small, $\sin \theta \approx \theta$. $a_2$ is small and can be assumed to be zero. The equation of motion then becomes

$$\ddot{\theta} + \left(1 + \frac{2m_2}{m_1}\right)\frac{g}{l} \theta = 0.$$