One Degree of Freedom, Harmonically Excited Vibrations

1 Forced Harmonic Vibration

A mechanical system is said to undergo forced vibration whenever external energy is supplied to the system during vibration. External energy can be from either an applied force or an imposed displacement excitation.

The applied force or displacement excitation may be harmonic, nonharmonic but periodic, nonperiodic, or random. Harmonic excitations are of the forms, for example,

\[ F(t) = F_0 e^{i(\omega t + \Phi)} , \]
\[ F(t) = F_0 \cos(\omega t + \Phi) , \]
\[ F(t) = F_0 \sin(\omega t + \Phi) , \]

where \( F_0 \) is the amplitude, \( \omega \) is the frequency, and \( \Phi \) is the phase angle usually taken to be zero.

Under a harmonic excitation, the response of the system will also be harmonic with the same frequency as the excitation frequency. If the frequency of the harmonic excitation is close to the system natural frequency, the beating phenomenon will happen. This condition, known as resonance, is to be avoided to prevent failure of the system.

Consider a system in Figure 1. The equation of motion is

\[ m\ddot{x} + c\dot{x} + kx = F(t) . \]

Its general solution is

\[ x(t) = x_h(t) + x_p(t) , \]

where \( x_h(t) \) is the homogeneous solution (the solution when \( F(t) = 0 \) as was studied in the free vibration) and \( x_p(t) \) is the particular solution.

Since the free-vibration response \( x_h(t) \) dies out with time under each of the three conditions of damping (underdamping, critical damping, and overdamping), the general solution eventually reduces to the particular solution \( x_p(t) \), which represents the steady-state vibration. Figure 2 shows homogeneous, particular, and general solutions for the underdamped case.
1.1 Undamped System under $F_0 \cos \omega t$

Consider the system in Figure 1 but without damper. If a force $F(t) = F_0 \cos \omega t$ acts on the mass $m$, the equation of motion is given by

$$m\ddot{x} + kx = F_0 \cos \omega t.$$  \hspace{1cm} (1)

The solution is

$$x(t) = x_h(t) + x_p(t),$$

where

$$x_h(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t,$$

$$x_p(t) = X \cos \omega t.$$  \hspace{1cm} (2)

Applying initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$, and by substituting (2) into (1), the three unknowns $C_1$, $C_2$, and $X$ can be solved, then the solution becomes

$$x(t) = \left( x_0 - \frac{F_0}{k - m\omega^2} \right) \cos \omega_n t + \left( \frac{x_0}{\omega_n} \right) \sin \omega_n t + \left( \frac{F_0}{k - m\omega^2} \right) \cos \omega t.$$  \hspace{1cm} (3)

Letting $\delta_{st} = F_0 / k$ denote the static deflection of the mass under a force $F_0$, we have

$$\frac{X}{\delta_{st}} = \frac{1}{1 - \left( \frac{\omega}{\omega_n} \right)^2}.$$  \hspace{1cm} (4)

Let the forcing frequency $\omega$ be slightly less than the natural frequency:

$$\omega_n - \omega = 2\varepsilon.$$  \hspace{1cm} (5)

Then $\omega_n \approx \omega$ we have

$$\omega + \omega_n \approx 2\omega.$$  \hspace{1cm} (6)

Using (5) and (6), (4) becomes

The total response (2) can also be written in three cases as follows:

1) For $\omega / \omega_n < 1$, we have

$$x(t) = A \cos (\omega_n t - \Phi + \frac{\delta_{st}}{1 - \left( \frac{\omega}{\omega_n} \right)^2} \cos \omega t).$$

2) For $\omega / \omega_n > 1$, we have

$$x(t) = A \cos (\omega_n t - \Phi - \frac{\delta_{st}}{1 - \left( \frac{\omega}{\omega_n} \right)^2} \cos \omega t).$$

3) For $\omega \approx \omega_n$, we have a beating phenomenon. Letting $x_0 = \dot{x}_0 = 0$, we have

$$x(t) = \left( \frac{F_0}{m} \right) \frac{1}{\omega_n^2 - \omega^2} \left( \cos \omega t - \cos \omega_n t \right) + \left( \frac{F_0}{m} \right) \frac{1}{\omega_n^2 - \omega^2} \left[ 2 \sin \frac{\omega + \omega_n}{2} t \cdot \sin \frac{\omega_n - \omega}{2} t \right].$$

The quantity $X / \delta_{st}$ represents the ratio of the dynamic to the static amplitude of motion and is called amplitude ratio.
Define period of beating to be \( \tau_b = \frac{2\pi}{2\varepsilon} = \frac{2\pi}{\left(\omega_n - \omega\right)} \). Define frequency of beating to be \( \omega_b = 2\varepsilon = \omega_n - \omega \).

The plots of the total responses of all three cases are given in Figure 3.
Example 1: [1] A reciprocating pump with mass 68 kg is mounted as shown below at the middle of a steel plate of thickness 1 cm, width 50 cm, and length 250 cm. During operation, the plate is subjected to a harmonic force \( F(t) = 220 \cos{62.832t} \) N. Find the amplitude of vibration of the plate.

Solution

The plate can be modeled as a fixed-fixed beam with equivalent spring constant

\[
k = \frac{192EI}{l^3}
\]

\[
= \frac{192 \left(200 \times 10^5\right) \left(\frac{1}{12} \left(50 \times 10^{-2}\right) \left(10^{-2}\right)^3\right)}{(250 \times 10^{-2})^3}
\]

\[
= 102400.82 \text{ N/m}
\]

From (3), we have

\[
X = \frac{F_0/k}{1 - \left(\frac{\omega}{\omega_n}\right)^2}
\]

\[
= \frac{220/102400.82}{1 - \left(\frac{62.832}{\sqrt{102400.82/68}}\right)^2}
\]

\[
= -0.001325 \text{ m}
\]

The negative sign indicates that the response \( x(t) \) is out of phase with the excitation \( F(t) \).
1.2 Damped System under $F_0 \cos \omega t$

For the damped system, the equation of motion becomes

$$mx + cx + kx = F_0 \cos \omega t.$$  \hspace{1cm} (7)

Assume the particular solution in the form

$$x_p(t) = X \cos(\omega t - \Phi),$$

where $X$ and $\Phi$ are unknown constants to be determined. Substituting into (7) and equating the coefficients of $\cos \omega t$ and $\sin \omega t$ on both sides, we obtain

$$X = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2 \omega^2}}^{1/2}$$

and

$$\Phi = \tan^{-1}\left(\frac{c \omega}{k - m\omega^2}\right).$$

Figure 4 shows plots of forcing function and particular solution.

Recall that $\omega_n = \sqrt{k/m}$ = undamped natural frequency, $\zeta = c / c_n = c / 2m\omega_n$, $\delta_n = F_0 / k$ = deflection under the static force $F_0$, and $r = \omega / \omega_n = \text{frequency ratio}$. The amplitude ratio and the phase angle is given by

$$\frac{X}{\delta_n} = \frac{1}{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^{1/2} + \left(2\zeta \frac{\omega}{\omega_n}\right)} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}},$$

and

$$\phi = \tan^{-1}\left(\frac{2\zeta \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right) = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right).$$

The plots of amplitude ratio and phase angle versus frequency ratio are given in Figure 5 and Figure 6 respectively.

The total response is given by

$$x(t) = x_h(t) + x_p(t).$$

For an underdamped system, we have

$$x(t) = X_0 e^{-\zeta \omega_n t} \cos(\omega_n t - \phi_0) + X \cos(\omega t - \phi).$$
where \( \omega_d = \sqrt{1 - \zeta^2 \omega_n} \). \( X_0 \) and \( \Phi_0 \) are unknown constants to be determined from initial conditions. For the initial conditions \( x(0) = x_0 \) and \( \dot{x}(0) = \dot{x}_0 \), we have two equations to solve for two unknowns

\[
x_0 = X_0 \cos \phi_0 + X \cos \phi,
\]
\[
\dot{x}_0 = -\zeta \omega X_0 \cos \phi_0 + \omega_d X_0 \sin \phi_0 + \omega X \sin \phi_0.
\]

\( \text{(9)} \)
Example 2: [1] Find the total response of a single degree of freedom system with \( m = 10 \) kg, \( c = 20 N \cdot s/m \), \( k = 4000 N/m \), \( \dot{x}_0 = 0 \), and \( x_0 = 0.01 \) under an external force \( F(t) = F_0 \cos \omega t \) with \( F_0 = 100 N \) and \( \omega = 10 \text{ rad/s} \).

Solution
From the data, we have 
\[
\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20 \text{ rad/s},
\]
\[
\delta_n = \frac{F_0}{k} = \frac{100}{4000} = 0.025 \text{ m},
\]
\[
\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{20}{2\sqrt{(4000)(10)}} = 0.05,
\]
\[
\omega_d = \sqrt{1-\zeta^2} \omega_n = \sqrt{1-(0.05)^2} (20) = 19.97 \text{ rad/s},
\]
\[
r = \frac{\omega}{\omega_n} = \frac{10}{20} = 0.5.
\]
\[
X = \frac{\delta_n}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{0.025}{\sqrt{(1-0.5^2)^2+(2\cdot0.5\cdot0.5)^2}} = 0.03326 \text{ m}.
\]
\[
\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{2\cdot0.5\cdot0.5}{1-0.5^2}\right) = 3.814075^\circ.
\]
Substituting the data above into (9), we get \( X_0 = 0.0233 \) and \( \Phi_0 = 5.587^\circ \).

For small values of damping, we can take
\[
\left(\frac{X}{\delta_n}\right)_{\text{max}} = \left(\frac{X}{\delta_n}\right)_{\omega=\omega_n} = \frac{1}{2\zeta} = Q.
\] (10)

The difference between the frequencies associated with the half power \((Q/\sqrt{2})\) points \( R_1 \) and \( R_2 \) is called bandwidth of the system.

![Figure 7: Harmonic response curve showing half power points and bandwidth.](image)

To find \( R_1 \) and \( R_2 \), we set \( X/\delta_n = Q/\sqrt{2} \) in (8) to obtain
\[
r^4 - r^2(2-4\zeta^2) + (1-8\zeta^2) = 0
\]
whose solutions are
\[\begin{align*}
r_1^2 &= R_1^2 = \left(\frac{\omega_1}{\omega_n}\right)^2 \approx 1 - 2\zeta, \quad r_2^2 &= R_2^2 = \left(\frac{\omega_2}{\omega_n}\right)^2 \approx 1 + 2\zeta.
\end{align*}\]

Using the relation \(\omega_2 + \omega_1 = 2\omega_n\) and
\(\omega_2^2 - \omega_1^2 = (\omega_2 + \omega_1)(\omega_2 - \omega_1) = (R_2^2 - R_1^2)\omega_n^2 \approx 4\zeta\omega_n^2\), we have that the bandwidth is given by
\[\Delta\omega = \omega_2 - \omega_1 \approx 2\zeta\omega_n.\]

Combining the bandwidth equation with (10), we obtain
\[Q \approx \frac{1}{2\zeta} \approx \frac{\omega_n}{\omega_2 - \omega_1}.\]

It can be seen that \(Q\) can be used for estimating the equivalent viscous damping and the natural frequency in mechanical systems.

**1.3 Damped System under \(F_0e^{i\omega t}\)**

Let the harmonic forcing function be represented in complex form as \(F(t) = F_0e^{i\omega t}\). The equation of motion becomes
\[m\ddot{x} + c\dot{x} + kx = F_0e^{i\omega t}.\] (11)

Assume the particular solution
\[x_p(t) = Xe^{i\omega t}.\]

Substituting into (11), we have
\[X = \frac{F_0}{(k - m\omega^2) + ic\omega} = F_0 \left[\frac{k - m\omega^2}{(k - m\omega^2)^2 + c^2\omega^2} - i\frac{c\omega}{(k - m\omega^2)^2 + c^2\omega^2}\right] = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} e^{-i\phi},\]

where
\[\phi = \tan^{-1}\left(-\frac{c\omega}{k - m\omega^2}\right).\]

Thus, the particular solution (or steady-state solution) becomes
\[x_p(t) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} e^{i(\omega t - \phi)}.\] (12)

The complex frequency response of the system is defined to be
\[H(i\omega) \equiv \frac{X}{F_0 / k} = \frac{1}{1 - r^2 + i2\zeta r}\]
whose magnitude is given by
\[|H(i\omega)| = \left|\frac{kX}{F_0}\right| = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}.\]

\(H(i\omega)\) can be used in the experimental determination of the system parameters \((m, c, \text{ and } k)\).
If \( F(t) = F_0 \cos \omega t \), the corresponding particular solution is the real part of (12), which is

\[
x_p(t) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \cos(\omega t - \phi).
\]

If \( F(t) = F_0 \sin \omega t \), the corresponding particular solution is the imaginary part of (12), which is

\[
x_p(t) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \sin(\omega t - \phi).
\]

2 Support Motion

Sometimes the base or support of a spring-mass-damper system undergoes harmonic motion, as shown in Figure 8.

![Figure 8: Base excitation.](image)

The equation of motion is given by

\[
mx\ddot{x} + c(x - \dot{y}) + k(x - y) = 0.
\]

Supposing that \( y(t) = Y \sin \omega t \), the equation of motion becomes

\[
m\ddot{x} + c\dot{x} + kx = ky + c\dot{y} = kY \sin \omega t + c\omega Y \cos \omega t = A \sin(\omega t - \alpha),
\]

where \( A = Y \sqrt{k^2 + (c\omega)^2} \) and \( \alpha = \tan^{-1}\left(-\frac{c\omega}{k}\right) \). This is similar to having the forcing function \( F(t) = F_0 \sin \omega t \) acting on the system and the same analysis as the previous section can be applied.

The particular solution is similar to (13) and is given by

\[
x_p(t) = X \sin(\omega t - \phi_1 - \alpha) = \frac{Y}{\sqrt{k^2 + (c\omega)^2}} \frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2},
\]

where \( \phi_1 = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) \) and

\[
\phi = \tan^{-1}\left[\frac{mc\omega^3}{k(k - m\omega^2) + (c\omega)^2}\right] = \tan^{-1}\left[\frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2}\right].
\]

2.1 Displacement Transmissibility

The ratio of the amplitude of the response \( x_p(t) \) to that of the base motion \( y(t) \), \( \frac{X}{Y} \), is called the displacement transmissibility. The ratio is given by

\[\text{transmissibility} = \frac{X}{Y} = \frac{\text{amplitude of response}}{\text{amplitude of base motion}}.\]
\[
\frac{X}{Y} = \left[ \frac{k^2 + (c\omega)^2}{(k - m\omega^2) + (c\omega)^2} \right]^{1/2} = \left[ \frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2}.
\]

The plots between \( \frac{X}{Y} \) and \( \phi \) versus frequency ratio \( r = \omega / \omega_n \) is given in Figure 9.

**2.2 Force Transmissibility**

Let \( F \) be the force transmitted to the base or support due to the reactions from the spring and the dashpot. We have

\[
F = k(x - y) + c(x - \dot{y}) = -m\ddot{x}.
\]

The steady-state solution \( x_p(t) \) was found to be \( x_p(t) = X \sin(\omega t - \phi) \). Therefore,

\[
F = m\omega^2 X \sin(\omega t - \phi) = F_T \sin(\omega t - \phi).
\]

\( F_T \) is called dynamic force amplitude. The ratio \( F_T / kY \) is called the force transmissibility and is given by
\[ F_T = r^2 \left[ \frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2}. \]

Figure 10 shows the force transmissibility. The force transmissibility concept is used in the design of vibration isolation systems.

The steady-state solution is given similar to (13) by

\[ z(t) = \frac{m\omega^2 Y \sin(\omega t - \phi_i)}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = \frac{Z \sin(\omega t - \phi_i)}{\sqrt{1 - r^2}}, \]

where

\[ Z = \frac{m\omega^2 Y}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = \frac{r^2}{\sqrt{1 - r^2}^2 + (2\zeta r)^2}, \]
\[ \phi_i = \tan^{-1} \left( \frac{c\omega}{k - m\omega^2} \right) = \tan^{-1} \left( \frac{2\zeta r}{1 - r^2} \right). \]

The ratio \( Z/Y \) is shown in Figure 11 and plot of \( \phi_i \) is in Figure 6.

**2.3 Relative Motion**

Let \( z = x - y \) denote the motion of the mass relative to the base. The equation of motion becomes

\[ m\ddot{z} + c\dot{z} + kz = -m\ddot{y} = m\omega^2 Y \sin \omega t. \]
Example 3: [1] Consider a simple model of a motor vehicle below. The vehicle has a mass of 1200 kg. The spring constant is 400 kN/m and the damping ratio of $\zeta = 0.5$. If the vehicle speed is 20 km/hr, determine the displacement amplitude of the vehicle. The road surface varies sinusoidally with an amplitude of $Y = 0.05 \text{ m}$ and a wavelength of 6 m.

Solution

From the given data, we can compute the following quantities:

$$\omega = 2\pi f = 2\pi \left( \frac{20 \times 1000}{3600} \right) \left( \frac{1}{6} \right) = 0.291 \text{ rad/s},$$

$$\omega_n = \sqrt{\frac{k}{m}} = \left( \frac{400 \times 10^3}{1200} \right)^{1/2} = 18.2574 \text{ rad/s},$$

$$r = \frac{\omega}{\omega_n} = \frac{5.81778}{18.2574} = 0.318653,$$

$$X = \frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}^{1/2}$$

$$= \frac{1 + (2 \times 0.5 \times 0.318653)^2}{(1 - 0.318653)^2 + (2 \times 0.5 \times 0.318653)^2}^{1/2},$$

$$X = 1.469237.$$

$$Y = 1.469237(0.05) = 0.073462 \text{ m}.$$
Example 4: [1] A heavy machine, weighing 3000 N, is supported on a resilient foundation. The foundation has spring stiffness \( k = 40,000 \, N/m \).
The machine vibrates with an amplitude of 1 cm when the base of the foundation is subjected to harmonic oscillation at the undamped natural frequency of the system with an amplitude of 0.25 cm. Find (a) the damping constant of the foundation, (b) the dynamic force amplitude on the base, and (c) the amplitude of the displacement of the machine relative to the base.

Solution

a) Since \( \omega = \omega_n \), we have \( r = 1 \). Therefore,

\[
\frac{X}{Y} = \frac{0.01}{0.0025} = 4 = \left[ 1 + \left( \frac{2\zeta}{\omega_n} \right)^2 \right]^{1/2}.
\]

We then have \( \zeta = 0.1291 \). The damping constant is given by

\[
c = \zeta \cdot c_n = \zeta \cdot 2\sqrt{km} = 903.05 \, N \cdot s/m.
\]

b) \( F_r = Yk \left[ \frac{1 + 4\zeta^2}{4\zeta^2} \right]^{1/2} = 400 \, N \).

c) \( Z = \frac{Y}{2\zeta} = \frac{0.0025}{2(0.1291)} = 0.00968 \, m \).

3 Rotating Unbalance

Consider a machine with rotating unbalanced masses in Figure 12.

Figure 12: A machine with rotating unbalanced masses.
The total mass of the machine is $M$, and there are two eccentric masses $m/2$ rotating in opposite directions with a constant angular velocity $\omega$. We consider two equal masses $m/2$ rotating in opposite directions in order to have the horizontal components of excitation of the two masses cancel each other.

The equation of motion is given by

$$M\ddot{x} + c\dot{x} + kx = me\omega^2 \sin \omega t.$$ 

The solution of this equation is similar to (13) and is given by

$$x_p(t) = X \sin(\omega t - \phi),$$

where

$$X = \frac{me\omega^2}{\left[(k - M\omega^2)^2 + (c\omega)^2\right]^{1/2}} r,$$

$$\phi = \tan^{-1}\left(\frac{c\omega}{k - M\omega^2}\right).$$

The plots between $MX/me$ and $\phi$ versus $r$ are given in Figure 11 and Figure 6.

Example 5: [1] The figure below depicts a Francis water turbine. Water flows from A into the blades B and down into the tail race C. The rotor has a mass of 250 kg and an unbalance (me) of 5 kg-mm. The radial clearance between the rotor and the stator is 5 mm. The turbine is to be operated at 6000 rpm. The steel shaft carrying the rotor can be assumed to be clamped at the bearings. Determine the diameter of the shaft so that the rotor is always clear of the stator. Assume damping is negligible.
Lesson 6 Homework Problems

Homework problems are from the required textbook *(Mechanical Vibrations*, by Singiresu S. Rao, Prentice Hall, 2004)

References

Solution
Setting \( c = 0 \), we have

\[
X = \frac{m\omega^2}{(k - M\omega^2)}
\]

\[
= \frac{m\omega^2}{k(1 - r^2)}
\]

\[
0.005 = \frac{(5.0 \times 10^{-3}) \times (200\pi)^2}{k \left[ 1 - \left( \frac{200\pi}{0.004k} \right)^2 \right]}
\]

\[
k = 10.04 \times 10^6 \pi^2 \text{ N/m}.
\]

Since for the steel beam, \( k = \frac{3EI}{l^3} = \frac{3E}{l^3} \left( \frac{\pi d^4}{64} \right) \), we have

\[
d^4 = \frac{64kl^3}{3\pi E} = \frac{(64)(10.04 \times 10^4 \pi^2)(2^3)}{3\pi(2.07 \times 10^{11})} = 2.6005 \times 10^{-4} \text{ m}^4 \text{ and}
\]

\[
d = 0.1270 \text{ m} = 127 \text{ mm}.
\]