General Information

Instructor:
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Lecture:
Monday 9.00 - 12.00 Room is to be determined.

Downloadable Class Notes:
Class notes for all lessons can be downloaded from my website above.

Required Textbook:

References:

Grade Breakdown:
Homeworks 50%
Quizzes 30%
In-Class Oral Questions 20%

Office Hours:
You are welcome to come into my office at any time.

Class Structure:
Three hours are divided into three sections of one hour. In each hour, the class structure is as follows:
50 minutes: Study new materials (or discuss homework due that day)
10 minutes: Break

Rules:
1) Quiz is an individual effort. You can use only your own notes and books during quizzes.
2) You cannot use a personal computer during quizzes. During quizzes, only graphical calculator is allowed (Texas Instruments TI-89 or compatible.)
3) Homework problems are from the required textbook [1]. Each homework is due at next class. Homework is a team effort. Teams of two to three people will be assigned during the first class. You may consult your friends or me, but copying from other team is strictly prohibited.
4) I do not accept late homework.
5) All in-class quizzes are random.
6) Grade sheet, which includes your score, ranking, and estimated grade, is posted on my website weekly.
7) Be punctual. If you are late, wait until next break to get in.
### Course Schedule:

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### Course Content:

Contents and homework problems are based on the required textbook [1].

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1 Nonlinear Models and Nonlinear Phenomena

- Linear versus nonlinear systems
  A linear system obeys principle of superposition shown in Figure 1. If the system does not obey principle of superposition, the system is said to be nonlinear system.

- Plant in nonlinear form
  We will deal with dynamical systems that are modeled by a finite number of coupled first-order ordinary differential equations

\[
\dot{x}_i = f_i(t, x_1, \ldots, x_n, u_1, \ldots, u_p), \\
\vdots \\
\dot{x}_n = f_n(t, x_1, \ldots, x_n, u_1, \ldots, u_p),
\]

where \( \dot{x}_i \) denotes the derivative of \( x_i \) with respect to time variable \( t \) and \( u_1, u_2, \ldots, u_p \) are specified input variables. \( x_1, x_2, \ldots, x_n \) are called state variables.
Let \( \mathbf{x} = [x_1, x_2, \ldots, x_n]^T \) be a column vector of the state variables, let \( \mathbf{u} = [u_1, u_2, \ldots, u_p]^T \) be a column vector of the input variables, and let
\[
f(t, x, u) = [f_1(t, x, u), f_2(t, x, u), \ldots, f_n(t, x, u)]^T
\]
be a column vector of functions, we have
\[
\dot{x} = f(t, x, u). \quad (1)
\]
Sometimes, another equation
\[
y = h(t, x, u) \quad (2)
\]
is used to define a \( q \)-dimensional output vector \( \mathbf{y} \). (1) and (2) together are called the nonlinear state-space model.

If \( \mathbf{u} = \gamma(t) \) or \( \mathbf{u} = \gamma(t, x) \), we can eliminate \( \mathbf{u} \) in (1) to have
\[
\dot{x} = f(t, x), \quad (3)
\]
which is called unforced state equation.

If \( f \) does not depend explicitly on \( t \), we have
\[
\dot{x} = f(x), \quad (4)
\]
which is called autonomous or time-invariant state equation.

- Equilibrium point
  A point \( x = x^* \) is said to be an equilibrium point if whenever the state of the system starts at \( x^* \), it will remain at \( x^* \) for all future time. Since \( x \) does not change at \( x^* \), we have that \( \dot{x} = 0 \), and therefore for an example of an autonomous system (4), we have
  \[
f(x^*) = 0.
\]

- A linear system

We can write a linear model in the form of (1) and (2) as
\[
\dot{x} = A(t)x + B(t)u,
\]
\[
y = C(t)x + D(t)u.
\]

Because linear systems are well understood, to analyze the nonlinear systems we often make use of the linearization process. By linearizing the nonlinear system around some local points, we can apply techniques for linear systems to nonlinear systems operating around these local points.

- Disadvantages of the linearized system
  1) It can only predict the behavior of the nonlinear system near some local points.
  2) It cannot predict the following nonlinear phenomena:
     - Finite escape time
       A nonlinear system's state can go to infinity in finite time; a linear system's state only goes to infinity as time approaches infinity.
     - Multiple isolated equilibria
       A linear system can have only one isolated equilibrium point; a nonlinear system can have more than one isolated equilibrium points.
     - Limit cycles
       A limit cycle is a stable oscillation of fixed amplitude and frequency. A nonlinear system can have a limit cycle irrespective of the initial state. A linear system cannot.
     - Subharmonic, harmonic, or almost-periodic oscillations
       A stable linear system under a periodic input produces an output of the same frequency. A nonlinear system under periodic excitation can oscillate with frequencies that are submultiples or multiples or the input frequency. It may even generate an almost-periodic oscillation.
     - Chaos
       A nonlinear system can have a more complicated steady-state behavior that is not equilibrium, periodic oscillation, or almost-periodic oscillation. These chaotic motions exhibit randomness, despite the deterministic nature of the system.
     - Multiple modes of behavior
       Two or more modes of behavior can be exhibited by the same nonlinear system. An unforced system may have more than one limit cycle. A forced system with periodic excitation may exhibit harmonic,
subharmonic, or more complicated steady-state behavior, depending on the amplitude and frequency of the input. It may even exhibit a discontinuous jump in the mode of behavior even if the amplitude or frequency of the excitation is smoothly changed.

In this course, we will only see the first three nonlinear phenomena.

2 Examples

2.1 Pendulum Equation

Example 1: [1] Consider a pendulum in Figure 2, let $k$ be a coefficient of friction, let the rod be rigid and has zero mass, let the ball have mass $m$, find equilibrium points of the system.

![Pendulum](image.png)

Figure 2: A pendulum.
Solution

From Newton’s second law, we have

\[ m l \ddot{\theta} = -mg \sin \theta - kl \dot{\theta}. \]

Let \( x_1 = \theta, \ x_2 = \dot{\theta} \), be the two state variables, we have

\[ \dot{x}_1 = x_2, \]
\[ \dot{x}_2 = -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2. \]

To find the equilibrium points, we set \( \dot{x}_1 = \dot{x}_2 = 0 \), we have

\[ 0 = x_2, \]
\[ 0 = -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2. \]

Therefore, the equilibrium points are located at \((x_1, x_2) = (n\pi, 0), \ n = 0, \pm 1, \pm 2, \ldots \) which are the two vertical positions.

2.2 Tunnel-Diode Circuit

♥ Example 2: [1] A tunnel-diode circuit is given in Figure 3. The tunnel diode is characterized by \( i_R = h(v_R) \). Find the equilibrium points of the system.
Solution

From the two facts,

\[ i_C = C \frac{dv_C}{dt} \quad \text{and} \quad v_L = L \frac{di_L}{dt}, \]

let \( x_1 = v_C \) and \( x_2 = i_L \) be the two state variables and let \( u = E \) be the system input, we then need to write \( i_C \) and \( v_L \) in terms of \( x_1, x_2, \) and \( u. \)

Using Kirchhoff's current law at node A, we have

\[ i_C + i_R - i_L = 0. \]

Therefore,

\[ i_C = -h(x_1) + x_2. \]

Using Kirchhoff's voltage law, we have

\[ v_C - E + Ri_L + v_L = 0. \]

Hence,

\[ v_L = -x_1 - Rx_2 + u. \]

We can now write the state equations as

\[ \dot{x}_1 = \frac{1}{C} \left[ -h(x_1) + x_2 \right], \]

\[ \dot{x}_2 = \frac{1}{L} \left[ -x_1 - Rx_2 + u \right]. \]

Finding the equilibrium points by setting \( \dot{x}_1 = \dot{x}_2 = 0, \) we have
Therefore, we have
\[ i_R = h(x_1) = \frac{E}{R} - \frac{1}{R} x_1, \]
which represents a straight line of the equilibrium points. It can be plotted as follows:

2.3 Mass-Spring System

Example 3: [1] Consider a mass-spring mechanical system shown in Figure 4. Assume that \( F_{sp} = g(y) \) and \( g(0) = 0 \).

The spring can be modeled as a linear spring, a softening spring, or a hardening spring as follows.

For linear spring,
\[ F_{sp} = ky. \]

For softening spring,
\[ F_{sp} = ky - ka^2 y^3. \]

For hardening spring,
\[ F_{sp} = ky + ka^2 y^3. \]

The friction force \( F_f \) may have components due to static, Coulomb, and viscous friction.

For viscous friction,
\[ F_f = c\dot{y}. \]

For Coulomb friction,
\[ F_f = \mu_s m g \text{ sign}(\dot{y}), \quad |\dot{y}| > 0. \]

For static friction,
\[ F_f = \begin{cases} -ky & \text{when } \dot{y} = 0, |\dot{y}| \leq \mu_s m g / k, \dot{y} = 0 \\ -\mu_s m g \text{ sign}(\dot{y}) & \text{when } \dot{y} = 0, |\dot{y}| > \mu_s m g / k, \dot{y} \neq 0 \end{cases}. \]

The \text{ sign}(x) \text{ is the signum function defined as}
The friction force $F_f$ can be plotted as Figure 5.

Let $x_1 = y$ and $x_2 = \dot{y}$ be the two state variables and $u = F$ be the system input, find the state model.
Solution

From Newton’s second law, we have
\[ m \ddot{y} = F - F_f - F_{sp}. \]

Since the friction force is a function of \( y \) and \( \dot{y} \), we can write
\[ F_f = F_f(y, \dot{y}) = F_f(x_1, x_2), \]
and the spring force is a function of \( y \), we can write
\[ F_{sp} = F_{sp}(y) = F_{sp}(x_1). \]

Therefore, we have the state equations
\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \frac{u}{m} - \frac{1}{m} F_f(x_1, x_2) - \frac{1}{m} F_{sp}(x_1).
\end{align*}
\]

The second state equation above has discontinuous right-hand side from the idealization we adopted in friction modeling. In fact, the transition from static to dynamic friction is proved to be continuous in [2]. However, the discontinuous idealization simplifies the analysis. For example, when \( x_2 > 0 \), we obtain a linear model
\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \frac{u}{m} - \frac{1}{m} (cx_2 + \mu_k mg) - \frac{1}{m} kx_1,
\end{align*}
\]
and when \( x_2 < 0 \), we obtain another linear model
\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \frac{u}{m} - \frac{1}{m} (cx_2 - \mu_k mg) - \frac{1}{m} kx_1.
\end{align*}
\]

Thus, in each region, we can predict the behavior of the system via linear analysis. This is the so-called piecewise linear analysis, where a system is represented by linear model in various regions of the state space.

2.4 Negative-Resistance Oscillator

♥ Example 4: [1] Consider a negative-resistance oscillator circuit shown in Figure 6.

![Negative-resistance oscillator circuit](image)

Figure 6: Negative-resistance oscillator circuit.

a) Let \( x_1 = v \) and \( x_2 = \dot{v} \) be state variables, find the state equations.
b) Let \( z_1 = i_c \) and \( z_2 = v_c \) be state variables, find the state equations.
c) Find the mapping \( z = T(x) \) and the inverse mapping \( x = T^{-1}(z) \).
Solution

Using Kirchhoff’s current law, we have

\[ i_C + i_L + i = 0. \]

Therefore, we have

\[ C \frac{dv}{dt} + \frac{1}{L} \int_{-\infty}^{t} v(s) ds + h(v) = 0. \]  \hspace{1cm} (5)

Differentiating once and multiplying by \( L \), we get

\[ CL \frac{d^2 v}{dt^2} + v + Lh'(v) \frac{dv}{dt} = 0. \]

Using \( x_1 = v \), \( x_2 = \dot{v} \), we have state equations as

\[ \dot{x}_1 = x_2, \]

\[ \dot{x}_2 = -\frac{x_1}{CL} - \frac{h'(x_1)}{C} x_2. \]

Equation (5) can be rewritten to

\[ C \frac{dv_C}{dt} + i_L + h(v_C) = 0. \]

Using \( z_1 = i_L \), \( z_2 = v_C \), we have state equations as

\[ \dot{z}_1 = \frac{1}{L} z_2, \]

\[ \dot{z}_2 = -\frac{1}{C} z_1 - \frac{1}{C} h(z_2). \]

Since \( x_1 = z_2 \) and \( x_2 = \frac{1}{C} z_1 - \frac{1}{C} h(z_2) \), we have the mapping from \( x \) to \( z \) as

\[ z = T(x) = \begin{bmatrix} -Cx_2 - h(x_1) \\ x_1 \end{bmatrix}, \]

and the inverse mapping from \( z \) to \( x \) as

\[ x = T^{-1}(z) = \begin{bmatrix} z_2 \\ -\frac{1}{C} z_1 - \frac{1}{C} h(z_2) \end{bmatrix}. \]

2.5 Artificial Neural Network

Consider a neural network model in Figure 7.

\[ v_i = g(u_i) \] is a sigmoid function shown in Figure 8. \( u_i \) and \( v_i \) are input and output voltages of the \( i^{th} \) amplifier. Let \( x_i = v_i, i = 1, 2, \ldots, n \) be state variables, find the state equations.
Figure 8: A sigmoid function.
Solution

Using Kirchhoff’s current law at the input node of the \( i \)th amplifier, we get

\[
C_i \frac{du_i}{dt} = \sum_j \left( \frac{1}{R_j} (\pm v_j - u_i) \right) - \frac{1}{\rho_i} u_i + I_i
\]

\[
= \sum_j \left( T_{ij} v_j \right) - \frac{1}{R_i} u_i + I_i,
\]

where \( T_{ij} \) is a signed conductance whose magnitude is \( 1/R_j \), and whose sign is determined by the choice of the positive or negative output of the \( j \)th amplifier and

\[
\rho_i = \frac{1}{R_i} + \sum_j \frac{1}{R_{ij}}.
\]

Since \( x_i = v_i, i = 1, \ldots, n, \) are the state variables, we can obtain the state equations by differentiating \( x_i \) with respect to time

\[
\dot{x}_i = \frac{dv_i}{dt} = \frac{dg_i(u_i)}{du_i} \frac{du_i}{dt}
\]

\[
= \frac{dg_i(u_i)}{du_i} \left[ \sum_j \left( T_{ij} x_j \right) - \frac{1}{R_i} u_i + I_i \right]
\]

\[
= h_i(x_i) \left[ \sum_j \left( T_{ij} x_j \right) - \frac{1}{R_i} g_i^{-1}(x_i) + I_i \right],
\]

\[
i = 1, \ldots, n, \text{ noting that we let } h_i(x_i) = \frac{dg_i(u_i)}{du_i} \bigg|_{u=g_i^{-1}(x_i)}.
\]

2.6 Adaptive Control

\( \bullet \) Example 6: [1] A first-order linear system is given by

\[
\dot{y}_p = a_p y_p + k_p u,
\]

where \( u \) is the control input, \( y_p \) is the measured output, \( a_p \) and \( k_p \) are unknown constant parameters. The sign of \( k_p \) is known.

We want this plant to follow the input-output behavior of a reference model

\[
\dot{y}_m = a_m y_m + k_m r.
\]

Ideally, we need

\[ u = \theta^*_1 r + \theta^*_2 y_p, \]

where

\[
\theta^*_1 = \frac{k_m}{k_p} \text{ and } \theta^*_2 = \frac{a_m - a_p}{k_p},
\]

since if we substitute (8) into (6) we would get (7).

However, because we don’t know \( a_p \) and \( k_p \), the control input \( u \) must be in the form

\[ u = \theta_1 r + \theta_2 y_p, \]

where \( \theta_1 \) is an estimate of \( \theta^*_1 \) and \( \theta_2 \) is an estimate of \( \theta^*_2 \).

The adaptation rules for \( \theta_1 \) and \( \theta_2 \) are devised based on the gradient algorithm to be
\[
\dot{\theta}_1 = -\gamma (y_p - y_m) r, \\
\dot{\theta}_2 = -\gamma (y_p - y_m) y_p.
\]

Choosing \( e_0 = y_p - y_m \), \( \phi_1 = \theta_1 - \theta_1^* \), and \( \phi_2 = \theta_2 - \theta_2^* \) to be the state variables, find the state equations.
Solution
From (7) and (9), we have
\[
\dot{y}_m = a_p y_m + k_p (\theta_1 r + \theta_2 y_m).
\]
From (6) and (9), we have
\[
\dot{y}_p = a_p y_p + k_p (\theta r + \theta_2 y_p).
\]
Subtracting the two equations above, we obtain the error equation
\[
\dot{e}_0 = \dot{y}_p - \dot{y}_m
= (a_p + k_p \theta_2^*) e_0 + k_p (\theta_1 - \theta_1^*) r + k_p (\theta_2 - \theta_2^*) y_p.
\]
Together with (10), we get the three state equations as
\[
\dot{e}_0 = a_m e_0 + k_p \phi_1 r + k_p \phi_2 (e_0 + y_m),
\]
\[
\dot{\phi}_1 = -\gamma e_0 r,
\]
\[
\dot{\phi}_2 = -\gamma e_0 (e_0 + y_m).
\]

2.7 Common Nonlinearities
This section lists five common nonlinearities. The first four nonlinearities are memoryless, that is, the output of the nonlinearity at any instant of time is determined uniquely by its input at that instant.

2.7.1. Relay
Electromechanical relays, thyristor circuits, and other switching devices can be modeled as relay nonlinearity as shown in Figure 9. Relay nonlinearity can be modeled by the signum function
\[
y = \text{sign}(u) = \begin{cases} 
1, & \text{if } u > 0 \\
0, & \text{if } u = 0 \\
-1, & \text{if } u < 0 
\end{cases}
\]

2.7.2. Saturation
Saturation nonlinearity is common in all practical amplifiers and motors. They are also used as limiters to restrict the range of a variable. The saturation nonlinearity is shown in Figure 10.

\[
y = \text{sat}(u) = \begin{cases} 
k \frac{u}{\delta}, & \text{if } |u| \leq \delta \\
k \text{sign}(u), & \text{if } |u| > \delta 
\end{cases}
\]

Figure 9: Relay.
Figure 10: Saturation.
2.7.3. Dead zone

Dead zone characteristic is typical of valves and some amplifiers at low input signals. Dead zone is shown in Figure 11.

![Figure 11: Dead zone.](image)

2.7.4. Quantization

Figure 12 shows quantization nonlinearity, which is typical in analog-to-digital conversion of signals.

![Figure 12: Quantization.](image)

2.7.5. Hysteresis

Hysteresis is the input-output characteristics that have memory, that is, the output at any instant of time depends on the whole history of the input. An example of hysteresis, a relay-with-hysteresis characteristic, is shown in Figure 13.

Another example of hysteresis nonlinearity is the backlash characteristic shown in Figure 14, which is common in gears.

![Figure 13: Relay with hysteresis.](image)

![Figure 14: Backlash hysteresis.](image)
Lesson 1 Homework Problems

1.1, 1.2, 1.5, 1.8, 1.9, 1.12, 1.13, 1.14, 1.19, 1.22

Homework problems are from the required textbook (Nonlinear Systems, by Hassan K. Khalil, Prentice Hall, 2002)

References